Problem, §2.3, p117. Prove that \( \lim_{x \to 0^+} \sqrt{x} \, e^{\sin \frac{\pi}{x}} = 0 \).

Solution. (without any commentary). Let \( g(x) = \frac{\sqrt{x}}{e} \) and let \( h(x) = e\sqrt{x} \). Since \(-1 \leq \sin \frac{\pi}{x} \leq 1\) for every \( x \) and since \( \sqrt{x} > 0 \) for every \( x > 0 \), we have

\[
g(x) \leq \sqrt{x} \, e^{\sin \frac{\pi}{x}} \leq h(x)
\]

for all \( x > 0 \). We also have

\[
\lim_{x \to 0^+} g(x) = 0 = \lim_{x \to 0^+} h(x)
\]

According to the Squeeze Theorem, we can conclude that \( \lim_{x \to 0^+} \sqrt{x} \, e^{\sin \frac{\pi}{x}} = 0 \), as required.

Solution. (with a fair bit of commentary/explanation) First make an initial analysis. As \( x \to 0^+ \), notice that \( \frac{\pi}{x} \to +\infty \). So the values of \( \sin \frac{\pi}{x} \) are unpredictable (could be 0, could be 1, could be -1, in fact could be any number between 1 and -1) for the relevant values of \( x \) (values of \( x \) that lie just a little above 0), except that they will never go higher than 1 or lower than -1. This unpredictability means that \( \lim_{x \to 0^+} \sin \frac{\pi}{x} \) does not exist. So we cannot simply use limit theorems and/or the known continuity of the square root function and the exponential function to find the limit we want.

So what to do? Notice that in our initial analysis we did have some information about the values for \( \sin \frac{\pi}{x} \). We can use this information to create a Squeeze-Theorem argument. The information that \( \sin \frac{\pi}{x} \leq 1 \) and the exponential function is increasing gives us

\[
e^{\sin \frac{\pi}{x}} \leq e^1 = e
\]

for all \( x \). Similarly, we have \(-1 \leq \sin \frac{\pi}{x} \) for all \( x \), so \( e^{-1} \leq e^{\sin \frac{\pi}{x}} \) or, in other words

\[
\frac{1}{e} \leq e^{\sin \frac{\pi}{x}}
\]

for all \( x \). Each of these displayed inequalities can be multiplied by \( \sqrt{x} \) without changing the sense of the inequality, since \( \sqrt{x} > 0 \) for all \( x > 0 \). In other words, we have

\[
\frac{\sqrt{x}}{e} \leq \sqrt{x} \, e^{\sin \frac{\pi}{x}} \leq e\sqrt{x}
\]

for all \( x > 0 \).

Now let \( g(x) = \frac{\sqrt{x}}{e} \) and let \( h(x) = e\sqrt{x} \). We have

\[
g(x) \leq \sqrt{x} \, e^{\sin \frac{\pi}{x}} \leq h(x)
\]

for all \( x > 0 \), and

\[
\lim_{x \to 0^+} g(x) = 0 = \lim_{x \to 0^+} h(x)
\]

According to the Squeeze Theorem, we can conclude that \( \lim_{x \to 0^+} \sqrt{x} \, e^{\sin \frac{\pi}{x}} = 0 \), as required.