Find the indicated limit. Use l’Hospital’s Rule where appropriate. Consider using more elementary methods. If l’Hospital’s Rule doesn’t apply, explain why.

**Problem 5, §4.5, p305.** \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} \)

**Solution.** This limit is of the form \( \frac{0}{0} \). L’Hospital’s Rule would apply, and this problem could be done that way, but it’s much simpler to use a little algebra. (Another alternative would be to recognize this expression as a derivative, but that’s probably more time consuming on this one.) We have

\[
\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x - 1)(x + 1)}{x + 1} = \lim_{x \to -1} (x - 1) = -2
\]

**Problem 9, §4.5, p305.** \( \lim_{x \to 0} \frac{\tan px}{\tan qx} \)

**Solution.** Since \( \tan 0 = 0 \) and the tangent function is continuous wherever it’s defined, this limit is of the form \( \frac{0}{0} \). Use l’Hospital, as follows:

\[
\lim_{x \to 0} \frac{\tan px}{\tan qx} = \lim_{x \to 0} \frac{p \sec^2 px}{q \sec^2 qx} = \frac{p \cdot 1}{q \cdot 1} = \frac{p}{q}
\]

In making this calculation we’ve used the continuity of the secant function at 0, as well as the fact that \( \sec 0 = 1 \).

**Problem 11, §4.5, p305.** \( \lim_{x \to 0^+} \frac{\ln x}{x} \)

**Solution.** Since \( \lim_{x \to 0^+} \ln x = -\infty \) and \( \lim_{x \to 0^+} x = 0 \), we can determine the limit from the form we already have. Notice, also, that the numerator is negative and the denominator is positive for \( x > 0 \). Putting this information together, we conclude that

\[
\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty
\]

L’Hospital’s Rule does not apply since we have neither \( \frac{0}{0} \) nor \( \frac{\infty}{\infty} \). In fact, using l’Hospital’s Rule would lead us to a faulty conclusion.
Problem 13, §4.5, p305. \( \lim_{t \to 0} \frac{5^t - 3^t}{t} \)

Solution. Both numerator and denominator have limit 0, so we may use l'Hospital. To do so, we'll need the formula for the derivative of \(a^t\) where \(a\) is a fixed positive constant (such as 5 or 3). Either figure it out by using the fact that \(a^t = e^{t \ln a}\) or look it up. Either way, find that \( \frac{d}{dt}a^t = a^t \ln a \). Thus

\[
\lim_{t \to 0} \frac{5^t - 3^t}{t} = \lim_{t \to 0} \frac{5^t \ln 5 - 3^t \ln 3}{1} = \ln 5 - \ln 3 = \ln \frac{5}{3}
\]

Problem 19, §4.5, p305. \( \lim_{x \to \infty} \frac{x}{\ln(1 + 2e^x)} \)

Solution. As \(x \to \infty\), we have \(2e^x \to \infty\), so \(\ln(1 + 2e^x) \to \infty\). Thus this limit is of the form \(\infty / \infty\), amenable to l'Hospital’s Rule. Applying l'Hospital and simplifying, we have

\[
\lim_{x \to \infty} \frac{x}{\ln(1 + 2e^x)} = \lim_{x \to \infty} \frac{1}{\frac{2e^x}{1+2e^x}} = \lim_{x \to \infty} \frac{1+2e^x}{2e^x}
\]

This new limit is also of the form \(\infty / \infty\). We can either use a little algebra (factor out \(2e^x\) from both numerator and denominator) or we can use l'Hospital again. Since I'm guessing you used l'Hospital, I'll do that as well. This gives

\[
\lim_{x \to \infty} \frac{1+2e^x}{2e^x} = \lim_{x \to \infty} \frac{2e^x}{2e^x} = 1
\]

Now combine these results to get the value of the original limit:

\[
\lim_{x \to \infty} \frac{x}{\ln(1 + 2e^x)} = \lim_{x \to \infty} \frac{1+2e^x}{2e^x} = 1
\]

Problem 23, §4.5, p305. \( \lim_{x \to \infty} e^{-x} \ln x \)

Solution. As \(x \to \infty\), \(e^{-x} \to 0\) and \(\ln x \to \infty\), so this limit is of the form \(0 \cdot \infty\). A little algebra gives

\[
\lim_{x \to \infty} e^{-x} \ln x = \lim_{x \to \infty} \frac{\ln x}{e^x}
\]

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and the limit on the right-hand side is of the form $\infty/\infty$. We can now apply l’Hospital to find
\[
\lim_{x \to \infty} e^{-x} \ln x = \lim_{x \to \infty} \frac{\ln x}{e^x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{e^x} = 0
\]
since, as $x \to \infty$, we have $\frac{1}{x} \to 0$ and $e^x \to \infty$.

**Problem 29, §4.5, p305.** $\lim_{x \to \infty} xe^{\frac{1}{x}} - x$

**Solution.** As $x \to \infty$, $\frac{1}{x} \to 0$, $e^{\frac{1}{x}} \to 1$, and $xe^{\frac{1}{x}} \to \infty$. Thus this limit is of the form $\infty - \infty$, which is indeterminate. As a first step, factor out the $x$:
\[
\lim_{x \to \infty} xe^{\frac{1}{x}} - x = \lim_{x \to \infty} x(e^{\frac{1}{x}} - 1)
\]
The new limit is of the form $\infty \cdot 0$, also indeterminate. Use a little more algebra to bring it into the form $\frac{0}{0}$, and then use l’Hospital:
\[
\lim_{x \to \infty} x(e^{\frac{1}{x}} - 1) = \lim_{x \to \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{(e^{\frac{1}{x}})(\frac{-1}{x^2})}{\frac{1}{x^2}} = \lim_{x \to \infty} e^{\frac{1}{x}} = e^0 = 1
\]