Problem 12, §4.5, p305. Find $\lim_{x \to \infty} \frac{\ln \ln x}{x}$.

**Solution.** As $x \to \infty$, $\ln x \to \infty$ and so $\ln(\ln x) \to \infty$ as well. The given limit is therefore of the form $\frac{\infty}{\infty}$, which is indeterminate. Since there’s no obvious way to rewrite, use l’Hospital’s Rule:

$$\lim_{x \to \infty} \frac{\ln(\ln x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = 0$$

Thus the function $y = x$ grows more rapidly than $y = \ln(\ln x)$, as $x \to \infty$.

Problem 22, §4.5, p305. Find $\lim_{x \to -\infty} x^2 e^x$.

**Solution.** The given limit is indeterminate of the form $\infty \cdot 0$. A little bit of algebra will bring it into the form $\frac{\infty}{\infty}$, and then we can use l’Hospital’s Rule. As it turns out, we’ll need to use l’Hospital twice, both times to the form $\frac{\infty}{\infty}$. Here are the details.

$$\lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

Problem 31, §4.5, p305. Find $\lim_{x \to 0^+} x \sin x$.

**Solution.** As $x \to 0^+$, $\sin x \to 0^+$, so this limit is of the form $0^0$. This form is indeterminate, as the zero in the base tries to push the whole expression to 0, while the zero in the exponent tries to push the whole expression to 1.

This is exactly the sort of limit where we use l’Hospital’s Rule in combination with the logarithm function and then “e it up”. In this case, we’ll need to do a little bit of algebra to bring $\ln(x^{\sin x})$ into the form of a quotient, and the quotient we get will be of the form $\frac{\infty}{\infty}$:

$$\lim_{x \to 0^+} \ln(x^{\sin x}) = \lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

This last limit is still of the form $\frac{\infty}{\infty}$. First rewrite it using trigonometric identities, and then we’ll analyze further.

$$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-\sin x \tan x}{x}$$
If you remember the special limit \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), then you can see that we’ll have

\[
(2) \quad \lim_{x \to 0^+} \frac{-\sin x \tan x}{x} = \lim_{x \to 0^+} \left( \frac{\sin x}{x} \right) \lim_{x \to 0^+} (-\tan x) = 1 \cdot 0 = 0
\]

Otherwise, use l’Hospital again to find

\[
(2') \quad \lim_{x \to 0^+} \frac{-\sin x \tan x}{x} = \lim_{x \to 0^+} \frac{-\sin x \sec^2 x - \cos x \tan x}{1} = 0
\]

Combine (1) with either (2) or (2’) to obtain

\[
\lim_{x \to 0^+} \ln(x^{\sin x}) = 0
\]

and “e it up” to find

\[
\lim_{x \to 0^+} x^{\sin x} = \lim_{x \to 0^+} e^{\ln(x^{\sin x})} = e^{\lim_{x \to 0^+} \ln(x^{\sin x})} = e^0 = 1.
\]

**Problem 33, §4.5, p305.** Find \( \lim_{x \to 0} (1 - 2x)^{\frac{1}{x}} \)

**Solution.** As \( x \to 0, \frac{1}{x} \to \pm \infty \), so this limit is of the form \( 1^{\pm \infty} \), which is indeterminate. In fact, when \( x \) is just a bit above 0, we’ll have a number just less than 1 raised to a huge power, and there’ll be a tug of war between an overall value of 1 (because the base is nearly 1) and 0 (because a number smaller than 1 is being raised to a huge power). When \( x \) is just a bit below 0, we’ll have a number just above 1 raised to a huge negative power, which also, in the end, gives a tug of war between 1 and 0.

Once again we use l’Hospital’s Rule in combination with the logarithm function and then “e it up”. In this case, we’ll be using l’Hospital on a quotient that is of the form \( \frac{0}{0} \). Here are the details.

\[
\lim_{x \to 0} \ln(1 - 2x)^{\frac{1}{x}} = \lim_{x \to 0} \frac{\ln(1 - 2x)}{x} = \lim_{x \to 0} \frac{-2}{1} = -2
\]

Now “e it up” to find that

\[
\lim_{x \to 0} (1 - 2x)^{\frac{1}{x}} = \lim_{x \to 0} e^{\ln((1 - 2x)^{\frac{1}{x}})} = e^{\lim_{x \to 0} \ln(1 - 2x)^{\frac{1}{x}}} = e^{-2}.
\]