iff: “iff” is an abbreviation for “if and only if”.

Geometric Series: The geometric series \( a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \) will converge if and only if \(|r| < 1\). If \(|r| < 1\), then the sum of the series will be \( \frac{a}{1 - r} \). In other words, \( a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1 - r} \) if \(|r| < 1\) and diverges otherwise.

Term Test for Divergence: If \( \lim_{n \to \infty} a_n \neq 0 \) then \( \sum_{n=1}^{\infty} a_n \) diverges. If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) has a chance to converge, but further testing is needed to decide whether it does or not.

Harmonic Series: \( \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \) so the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

**Problem 9, §8.2, p580.** Let \( a_n = \frac{2n}{3n+1} \).

(a) Determine whether the sequence \( \{a_n\} \) is convergent.

(b) Determine whether the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

**Solution.**

(a) Since \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n}{3n+1} = \lim_{n \to \infty} \frac{2}{3 + \frac{1}{n}} = \frac{2}{3} \), the sequence \( \{a_n\} \) converges (to 3).

(b) From part (a), we know \( \lim_{n \to \infty} a_n = 3 \neq 0 \). According to the Term Test, the series \( \sum_{n=1}^{\infty} a_n \) diverges. (We can also say \( \sum_{n=1}^{\infty} a_n = +\infty \) in this case.)

Determine whether the series converges or diverges. If it converges, find its sum.

**Problem 13, §8.2, p580.** \( \sum_{n=1}^{\infty} 5 \left( \frac{2}{3} \right)^{n-1} \)

**Solution.** The given series \( \sum_{n=1}^{\infty} 5 \left( \frac{2}{3} \right)^{n-1} = 5 + 5 \cdot \frac{2}{3} + 5 \cdot \frac{2^2}{3^2} + 5 \cdot \frac{2^3}{3^3} + \cdots \)

is geometric, with initial term \( a = 5 \) and common ratio \( r = \frac{2}{3} \). Since \(|r| = \frac{2}{3} < 1\), the series
converges. For the sum, we have
\[ \sum_{n=1}^{\infty} 5 \left( \frac{2}{3} \right)^{n-1} = \frac{5}{1 - \frac{2}{3}} = 15 \]

Problem 17, §8.2, p580. \[ \sum_{n=1}^{\infty} \frac{n}{n + 5} \]

Solution. Since \( \lim_{n \to \infty} \frac{n}{n + 5} = \lim_{n \to \infty} \frac{1}{1 + \frac{5}{n}} = 1 \neq 0 \), the Term Test shows that the given series \( \sum_{n=1}^{\infty} \frac{n}{n + 5} \) diverges. In fact, \( \sum_{n=1}^{\infty} \frac{n}{n + 5} = +\infty \).

Problem 18, §8.2, p580. \[ \sum_{n=1}^{\infty} \frac{3}{n} \]

Solution. The given series is a variation on the harmonic series. According to Theorem 8(i) p579, if the given series did converge, then the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \left( \frac{1}{3} \cdot \frac{3}{n} \right) \) would also converge. We know that the harmonic series diverges, so the given series must also diverge. In fact \( \sum_{n=1}^{\infty} \frac{3}{n} = +\infty \).

Problem 26, §8.2, p580. \[ \sum_{n=1}^{\infty} \frac{1}{5 + 2^{-n}} \]

Solution. Since \( \lim_{n \to \infty} \frac{1}{5 + 2^{-n}} = \lim_{n \to \infty} \frac{1}{5 + \frac{1}{2^n}} = \frac{1}{5} \neq 0 \), the Term Test shows that the given series diverges: \( \sum_{n=1}^{\infty} \frac{1}{5 + 2^{-n}} = +\infty \).

Problem 28, §8.2, p580. \[ \sum_{n=1}^{\infty} \ln \frac{n}{n + 1} \] (See the hint on the HWK 21 assignment page.)

Solution. The Term Test is of no use here, since \( \lim_{n \to \infty} \ln \frac{n}{n + 1} = \ln 1 = 0 \). Following the hint,
examine the partial sums $S_n$. We have

$$S_n = \ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \cdots + \ln \frac{n}{n+1} = \ln \frac{1}{n+1}$$

Therefore

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{n+1} = -\infty.$$ By definition, then, the given series diverges. In fact

$$\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = -\infty.$$

**Problem 35, §8.2, p580.** Find the values of $x$ for which the series $\sum_{n=0}^{\infty} \frac{1}{x^n}$ converges. Find the sum of the series for those values of $x$ for which it converges.

**Solution.** The given series

$$\sum_{n=0}^{\infty} \frac{1}{x^n} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \cdots$$

is geometric, with initial term $a = 1$ and common ratio $r = \frac{1}{x}$. It therefore converges iff $|r| = \left| \frac{1}{x} \right| < 1$. Since $\left| \frac{1}{x} \right| < 1$ iff $|x| > 1$, the given series converges iff $|x| > 1$. In other words, it converges if $x < -1$, it diverges if $-1 \leq x \leq 1$, and it converges if $x > 1$. When $|x| > 1$, the series sum is

$$\sum_{n=0}^{\infty} \frac{1}{x^n} = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}$$

**Problem 40, §8.2, p580.** If the $n$th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = 3 - n2^{-n} = 3 - \frac{n}{2^n}$$

find $a_n$ and $\sum_{n=1}^{\infty} a_n$.

**Solution.** In preparation for finding $\lim_{n \to \infty} S_n$, notice that $\lim_{n \to \infty} \frac{x}{2^n} = \lim_{n \to \infty} \frac{1}{2^n \ln x} = 0$, by l'Hôpital’s Rule for the case $\frac{\infty}{\infty}$. Therefore $\lim_{n \to \infty} \frac{n}{2^n} = 0$, as well. This gives

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 3 - \frac{n}{2^n} \right) = 3 - 0 = 3$$
For $a_1$, we have

$$a_1 = S_1 = 3 - \frac{1}{2} = \frac{5}{2}$$

For $n > 1$, we have

$$a_n = S_n - S_{n-1} = 3 - \frac{n}{2^n} - \left(3 - \frac{n-1}{2^{n-1}}\right) = \frac{n-1}{2^{n-1}} - \frac{n}{2^n} = \frac{2(n-1) - n}{2^n} = \frac{n-2}{2^n}$$

Problem 48, §8.2, p580. Suppose that $\sum_{n=1}^{\infty} a_n$ (with $a_n \neq 0$) is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a divergent series. (See the hint on the HWK 21 assignment page.)

Solution. We’ve supposed that $\sum_{n=1}^{\infty} a_n$ converges. According to the Term Test (more precisely, according to Theorem 6 p578, which gives us the Term Test), we must have $\lim_{n \to \infty} a_n = 0$. Therefore $\lim_{n \to \infty} \left| \frac{1}{a_n} \right| = +\infty$ and so $\lim_{n \to \infty} \frac{1}{a_n} \neq 0$. Applying the Term Test, we see that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ must diverge.

Problem 50, §8.2, p580. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both divergent, is $\sum_{n=1}^{\infty} (a_n + b_n)$ necessarily divergent? (See the hint on the HWK 21 assignment page.)

Solution. No, it could happen that both of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge but that the terms somehow cancel each other out enough so that $\sum_{n=1}^{\infty} (a_n + b_n)$ converges. One example would be to have $\sum_{n=1}^{\infty} a_n$ be the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and to have $\sum_{n=1}^{\infty} b_n$ be the series $\sum_{n=1}^{\infty} \frac{1}{n}$. In this case, $\sum_{n=1}^{\infty} (a_n + b_n)$ would be $0 + 0 + 0 + \cdots + 0 + \cdots$, which obviously converges to 0. Another example would be $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-n)$. Theorem 8(ii) only tells us that the term-by-term sum of two convergent series is a convergent series. It tells us nothing about the term-by-term sum of two divergent series, which could converge or diverge.