Problem 3, §5.6, p401. Find \( \int xe^{2x} \, dx \).

Solution. Use integration by parts with \( u = x \) and \( dv = e^{2x} \, dx \). This gives \( du = dx \) and \( v = \frac{1}{2}e^{2x} \), and the IP formula
\[
\int u \, dv = uv - \int v \, du
\]
gives
\[
\int xe^{2x} \, dx = \frac{1}{2}xe^{2x} - \frac{1}{2} \int e^{2x} \, dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C
\]

Problem 7, §5.6, p401. Find \( \int x^2 \cos 3x \, dx \).

Solution. Begin with integration by parts, using \( u = x^2 \) and \( dv = \cos 3x \, dx \). This choice gives \( du = 2x \, dx \) and \( v = \frac{1}{3} \sin 3x \), so IP gives
\[
\int x^2 \cos 3x \, dx = \frac{1}{3}x^2 \sin 3x - \frac{2}{3} \int x \sin 3x \, dx
\]
Use a second integration by parts on the remaining integral, say with \( U = x \) and \( dV = \sin 3x \, dx \). This choice gives \( dU = dx \) and \( V = -\frac{1}{3} \cos 3x \) so the second IP gives us
\[
\int x^2 \cos 3x \, dx = \frac{1}{3}x^2 \sin 3x - \frac{2}{3} \left[ -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx \right]
= \frac{1}{3}x^2 \sin 3x + \frac{2x}{9} \cos 3x - \frac{2}{9} \int \cos 3x \, dx
= \frac{1}{3}x^2 \sin 3x + \frac{2x}{9} \cos 3x - \frac{2}{27} \sin 3x + C
\]

Problem 11, §5.6, p401. Find \( \int r^3 \ln r \, dr \).

Solution. In this case it works best to use IP with \( u = \ln r \) and \( dv = r^3 \, dr \). Then \( du = \frac{dr}{r} \) and \( v = \frac{1}{4}r^4 \). According to the IP formula,
\[
\int r^3 \ln r \, dr = \frac{1}{4}r^4 \ln r - \frac{1}{4} \int r^3 \, dr = \frac{1}{4}r^4 \ln r - \frac{1}{16}r^4 + C
\]
Problem 15, §5.6, p401. Find $\int_0^1 te^{-t} dt$.

Solution. Begin with integration by parts, using $u = t$ and $dv = e^{-t}$. Then $du = dt$ and $v = -e^{-t}$, so

$$\int_0^1 te^{-t} dt = \left[-te^{-t}\right]_0^1 + \int_0^1 e^{-t} dt = -e^{-1} + \left[-e^{-t}\right]_0^1 = -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = 1 - \frac{2}{e}$$

Problem 19, §5.6, p401. Find $\int_0^\frac{1}{2} \arcsin x \, dx$.

Solution. I’ll do the indefinite integral first this time. Use integration by parts with $u = \arcsin x$ and $dv = dx$. This gives $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$, so

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x \, dx}{\sqrt{1-x^2}}$$

This last integral can be found by substitution. Use $w = 1 - x^2$, which gives $dw = -2x \, dx$ and consequently

$$-\int \frac{x \, dx}{\sqrt{1-x^2}} = \frac{1}{2} \int w^{-\frac{1}{2}} \, dw = \frac{1}{2} w^{\frac{1}{2}} + C = \sqrt{1-x^2} + C$$

Putting these pieces together, we find

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x \, dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C$$

Before we compute the definite integral, recall that $\arcsin 0 = 0$ and $\arcsin \frac{1}{2} = \frac{\pi}{6}$. The definite integral is therefore

$$\int_0^{\frac{1}{2}} \arcsin x \, dx = \left[x \arcsin x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{1}{2} \frac{\pi}{6} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{1}{12} \left(\pi + 6\sqrt{3} - 12\right)$$
**Problem 23, §5.6, p401.** Find $\int_{\pi/6}^{\pi/2} \cos \theta \ln(\sin \theta) \, d\theta$.

**Solution.** My first inclination, which does work, would be to start with the substitution $w = \sin \theta$. However, it actually turns out to be easier to use integration by parts from the start with $u = \ln(\sin \theta)$ and $dv = \cos \theta \, d\theta$. This choice gives $du = \frac{\cos \theta}{\sin \theta}$ and $v = \sin \theta$. For the indefinite integral, integration by parts then gives

$$
\int \cos \theta \ln(\sin \theta) \, d\theta = \sin \theta \ln(\sin \theta) - \int \cos \theta \, d\theta = \sin \theta \ln(\sin \theta) - \sin \theta + C
$$

Therefore

$$
\int_{\pi/6}^{\pi/2} \cos \theta \ln(\sin \theta) \, d\theta = \left[ \sin \theta \ln(\sin \theta) - \sin \theta \right]_{\pi/6}^{\pi/2} = \left[ 0 - 1 \right] - \left[ \frac{1}{2} \ln \left( \frac{1}{2} \right) - \frac{1}{2} \right] = \frac{1}{2} \ln 2 - \frac{1}{2} = \frac{1}{2} (\ln 2 - 1)
$$
**Problem 38, §5.6, p401.** Prove/verify the reduction formula

\[ \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \]

**Solution.** Begin with integration by parts, using \( u = x^n \) and \( dv = e^x \, dx \). This choice gives \( du = nx^{n-1} \) and \( v = e^x \). Therefore integration by parts gives

\[ \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \]

which is the required formula.

**Problem 40, §5.6, p401.** Use the reduction formula from Problem 38 to find \( \int x^4 e^x \, dx \).

**Solution.** The hardest part about this problem is keeping track of the constant factors that accumulate as we use the reduction formula more and more times, in order to eventually eliminate the power of \( x \) in the integrand. Here’s one way to do that. Begin with the given integral and apply the reduction formula four times, first with \( n = 4 \), then \( n = 3 \), \( n = 2 \), \( n = 1 \), simplifying as we go, and then finish off the integral that’s left at the end. This gives

\[
\int x^4 e^x \, dx = x^4 e^x - 4 \int x^3 e^x \, dx \\
= x^4 e^x - 4[x^3 e^x - 3 \int x^2 e^x \, dx] \\
= x^4 e^x - 4x^3 e^x + 12 \int x^2 e^x \, dx \\
= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \int x e^x \, dx \\
= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 \int e^x \, dx \\
= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C \\
= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C
\]