1. Some of you may be familiar with a memorial known as the St. Louis Gateway Arch, in St. Louis, Missouri. This arch is in the shape of a catenary, actually an upside-down catenary, a curve that has an interesting history in mathematics. If one takes a length of chain (it should be perfectly flexible, not stiff, and it should have uniform density) and suspends it from its two ends (so that the two ends are level with each other and gravity acts on the chain to draw the middle of the chain downward), then the shape that the chain assumes is known as a catenary (from the Latin word for “chain”). A catenary looks a bit like a parabola—in fact Galileo conjectured that it was a parabola—but it was proved in the late 1600’s that the shape is not a parabola (which, as you probably remember, is the locus of points that are equidistant from a fixed point and a fixed line).

(a) St. Andrews University has an extensive web site on the history of mathematics, called the MacTutor History of Mathematics Archive. Browse the site (which you can reach from http://www-history.mcs.st-and.ac.uk), and other sources if you like, to learn a little about the catenary and its history. In particular, find out who the Bernoulli brothers were and what roles they played in the mathematical history of the catenary. (While you’re at it, you might like to learn about what Johann Bernoulli had to do with l’Hospital’s rule, which is a technique we’ll be studying a little later in the semester.) Write a few sentences to summarize what you found in this regard.

(b) Then write two or three paragraphs to summarize what else you learned that was most interesting to you and why.

2. This problem is related to the preceding one. The physics of the catenary can be used to derive an equation for the catenary, which turns out to be

\[ y = \frac{1}{2k} (e^{kx} + e^{-kx}) \]

Here the letter \( k \) denotes a constant that depends on various physical characteristics of the particular chain.

(a) For \( x \) restricted so that \(-1 \leq x \leq 1\), set up a definite integral whose value is the length of the corresponding portion of the catenary.

(b) Verify that \( 1 + (\frac{e^{kx} - e^{-kx}}{2})^2 \) can be rewritten as a perfect square. [Hint: start by expanding. Expect something nice to happen.]

(c) Now find the catenary length by evaluating the integral you found in (a).
3. The square bounded by the coordinate axes and the lines \( x = 2 \) and \( y = 2 \) is cut into two parts by the curve \( y^2 = 2x \). Show that these parts general equal volumes when revolved about the \( x \)-axis.

4. Consider a loaf of bread to be a hemisphere of radius \( r \), sliced into \( n \) pieces of equal width by parallel planes.
   
   (a) Ignoring the horizontal crust on the bottom, calculate the area of the crust of a typical slice.
   
   (b) Which slice has the most crust?

5. The average value of a function \( f \) over the interval \([a, b]\) is defined to be

\[
\frac{1}{b - a} \int_a^b f(x) \, dx
\]

   (a) Find the average values of \( f(x) = x \), \( f(x) = x^2 \), and \( f(x) = x^3 \) over the interval \([0, 1]\).

   (b) For an unspecified integer \( n \geq 1 \), what is the average value of \( f(x) = x^n \) over the interval \([0, 1]\)? (Make a guess, based on the evidence from part (a), but also verify your guess.)

   (c) What does the answer to part (b) imply about the average value of \( f(x) = x^n \), over the interval \([0, 1]\), as \( n \) gets larger and larger? How could you use the graph of \( f(x) = x^n \) to explain this?