Mathematics 120
Solutions for HWK 25a
- probs using Taylor remainder formula -

19. p686. For approximately what values of x can one replace $\sin x$ by $x - \frac{x^3}{6}$ with an error or magnitude no greater than 5×10^{-4}?

Before we start, notice that the suggested approximation function, $x - \frac{x^3}{6}$, is both the 3rd-order Maclaurin polynomial and the 4th-order Maclaurin polynomial for $\sin x$. We'll get better information if we think of $x - \frac{x^3}{6}$ as a 4th-order Taylor polynomial rather than 3rd and use the remainder formula for $R_4(x)$. So we'll need the 5th derivative for the $\sin x$, which is $\cos x$. With this in mind, we have

$$\sin x = x - \frac{x^3}{6} + R_4(x)$$

with

$$R_4(x) = \frac{\cos c}{5!} x^5$$

for some c between 0 and x. This gives $|R_4(x)| = |\frac{\cos c}{5!} x^5| \le \frac{|x|^5}{120}$, so we ask: for which values of x can we say $\frac{|x|^5}{120} \le 5 \times 10^{-4}$? Solve for $|x|$ as follows: $|x|^5 \le 600 \times 10^{-4} = 6 \times 10^{-2} = 0.06$, so $|x| \le (0.06)^{\frac{1}{5}} \approx 0.56968$.

20. p686. If $\cos x$ is replaced by $1 - \frac{x^2}{2}$ and $|x| < 0.5$, what estimate came be made of the error? Does $1 - \frac{x^2}{2}$ tend to be too large, or too small?

Let's answer the second question first. Notice that the Maclaurin series for $\cos x$, whose first two nonzero terms give the approximation $1 - \frac{x^2}{2}$ that we're to use here, is an alternating series, irrespective of whether x is positive or negative. For small x the factorials in the denominator will dominate the powers of x in the numerator, so the terms will definitely decrease in magnitude. And of course they tend to 0, since we know the cosine series converges for every x. Thus the conditions for the Alternating Series Test and Alternating Series Error Estimate are met. Since the first term we don't use is positive, the Alternating Series Error Estimate tells us that the approximation we are using undershoots $\cos x$. In other words, $1 - \frac{x^2}{2}$ tends to be too small, not too large.

Now for the first question. We could continue in the vein of using the Alternating Series Error Estimate, but since this HWK was largely designed to give you practice using the Taylor Remainder Formula, let's tackle the problem that way instead. Again, note that $1 - \frac{x^2}{2}$ is both a 2nd-order and a 3rd-order Maclaurin polynomial for $\cos x$. We'll view it as a 3rd-order Taylor polynomial and use the Taylor formula to estimate the corresponding remainder $R_3(x)$. Noting in advance that the fourth derivative of $\cos x$ is $\cos x$, we have

$$\cos x = 1 - \frac{x^2}{2} + R_3(x)$$

with

$$R_3(x) = \frac{\cos c}{4!} x^4$$

for some c between 0 and x. Since $|\cos c| \leq 1$ and $|x| < 0.5$, this gives $|R_3(x)| \leq \frac{1}{24}(0.5)^4 = \frac{1}{24 \cdot 16} = \frac{1}{384} \approx 0.002604167$.

23. p686. The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

What the text calls the Remainder Estimation Theorem is a consequence of the Taylor Remainder Formula that we've been using. It's pretty straightforward, but we didn't actually discuss it, so let me change the instructions on this problem slightly. Instead of "use the Remainder Estimation Theorem", let the problem read "use the Taylor Remainder Formula" to estimate etc.
We recognize the specified approximation as the 2nd-order Maclaurin polynomial for e^x. So we want to get an upper bound for $R_3(x)$ given that $|x| < 0.1$. The derivative we need is just e^x. We have

$$e^x = 1 + x + \frac{x^2}{2} + R_3(x)$$

with

$$R_3(x) = \frac{e^c}{3!}x^3$$

for some c between 0 and x. For $|x| < 0.1$, the largest value of e^c will occur when $x = 0.1$, and this largest value is $e^{0.1} \approx 1.105171$. Therefore, for $|x| < 0.1$, we have $R_3(x)| \leq \frac{1.1052}{6}(0.1)^3 \approx .000184195 \approx 1.84 \times 10^{-4}$. (The final answer in the text appears to be off by a decimal place, although the intermediate result is correct. Also, the text used a calulator value for $3^{0.1}$ instead of a calculator value for $e^{0.1}$ as an upper bound for the numerator e^c. Either one is correct, but ours gives a slightly tighter estimate.)