1. For each of the following limit statements (about a sequence), give the formal definition.

   (a) $\lim_{n\to\infty} a_n = L$ (L a real number)

For each positive number ϵ, there exists a positive integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

   (b) $\lim_{n\to\infty} a_n = +\infty$

For each positive number $B > 0$, there exists a positive integer N such that if $n > N$, then $a_n > B$.

2. Do (a) or (b), not both. For either (a) or (b) give a proof showing that the given statement is true. Your proof should be written in complete sentences and should be one that is based on the relevant formal definition, not on limit theorems or known limits.

   (a) $\lim_{n\to\infty} \frac{3n-11}{n} = 3$

Here's one possible proof (without the scratch paper).
Let $\epsilon > 0$ be given.
Choose N to be the first (or any) positive integer that satisfies $N \geq \frac{11}{\epsilon}$.
Suppose $n > N$.
Then $|\frac{3n-11}{n} - 3| = |\frac{-11}{n}| = \frac{11}{n} < \frac{11}{N} < \frac{11}{(\frac{11}{\epsilon})} = \epsilon$.

   (b) $\lim_{n\to\infty} (3n-1) = +\infty$

A possible proof, without scratch paper.
Let $B > 0$ be given.
Choose N to be the first (or any) positive integer that satisfies $N > B + 1$.
Suppose $n > N$.
Then $3n - 1 > 3N - 1 \geq 3(B+1) - 1 = 3B + 2 > B$.

(Another choice for N that could give a valid proof would be to use N satisfying $N \geq \frac{B+1}{3}$. That's the N you'd end up with if you begin by solving the inequality $3n - 1 > B$ for n in terms of B.)

3. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series whose partial sums are given by $S_n = \dfrac{10n}{n+2}$.

(a) Find $\sum_{n=1}^{98} a_n$.

$$\sum_{n=1}^{98} a_n = S_{98} = \frac{(10)(98)}{98+2} = \frac{980}{100} = 9.8$$

(b) Find a_{10}.

$$a_{10} = S_{10} - S_9 = \frac{100}{12} - \frac{90}{11} = \frac{5}{33}$$

(c) Use the definition of series convergence to verify that the given series converges.

The series will converge if and only iff the sequence $S_1, S_2, S_3, \cdots, S_n, \cdots$ has a finite limit. Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{10n}{n+2} = 10$, the series converges.

(d) Find the sum of the given series.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 10$$

4. (a) Find $\lim_{n\to\infty} \frac{\cos n + 3n}{n}$. Then use the Sandwich Theorem to prove that your answer is correct.

The specified limit is 3. Here's a Sandwich-Theorem proof that this is correct.
First, notice that

$$\frac{\cos n + 3n}{n} = 3 + \frac{\cos n}{n}$$

for every n. Also, for every n, we have $-1 \leq \cos n \leq 1$ and so $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$, which gives

$$3 + (\frac{-1}{n}) \leq 3 + \frac{\cos n}{n} \leq 3 + \frac{1}{n}$$

for every n. Putting these two observations together, we get

$$3 + (\frac{-1}{n}) \leq \frac{\cos n + 3n}{n} \leq 3 + \frac{1}{n}$$

for every n. Moreover,

$$\lim_{n\to\infty} (3 + (\frac{-1}{n})) = 3$$

and

$$\lim_{n\to\infty} (3 + (\frac{1}{n})) = 3$$

By the Sandwich Theorem,

$$\lim_{n\to\infty} \frac{\cos n + 3n}{n} = 3$$

(b) Does the sequence with general term $a_n = \sin(\frac{(4n+3)\pi}{2})$ converge? Why or why not?

The terms of the sequence are $\sin(\frac{7\pi}{2}), \sin(\frac{11\pi}{2}), \sin(\frac{15\pi}{2}), \cdots, \sin(2n\pi + \frac{3\pi}{2}), \cdots$. All of these terms are -1. In other words, the given sequence is the same as the constant sequence $-1, -1, -1, \cdots, -1, \cdots$, and therefore converges to -1. So, yes, the given sequence converges.

(c) Find $\lim\limits_{n\to\infty}(\frac{5+n}{3+n})^n$

This limit is of the form 1^∞. We replace n by x (at least in our thinking), use l'Hospital's rule to find $\lim \ln((\frac{5+n}{3+n})^n)$, and then e it up.

$$\begin{aligned}
\lim_{x\to\infty} \ln((\frac{5+x}{3+x})^x) &= \lim_{x\to\infty} x\ln(\frac{5+x}{3+x}) \\
&= \lim_{x\to\infty} \frac{\ln(\frac{5+x}{3+x})}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{(\frac{3+x}{5+x})(\frac{-2}{(3+x)^2})}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} (\frac{3+x}{5+x})(\frac{2x^2}{(3+x)^2}) \\
&= (1)(2) \\
&= 2
\end{aligned}$$

Therefore

$$\lim_{x\to\infty}(\frac{5+x}{3+x})^x = \mathrm{e}^2$$

which implies that

$$\lim_{n\to\infty}(\frac{5+n}{3+n})^n = \mathrm{e}^2$$

as well.