Context: Given a function $z = f(x, y)$ that has continuous first-order, second-order, and third-order partial derivatives. Given a critical point $a$ for $f$. (In other words, $a = (x_0, y_0)$ is a point at which both first-order partial derivatives of $f$ are zero.) We are testing the critical point $a$ to decide whether $f(x, y)$ has local maximum, local minimum, or neither at $a$.

To apply the Second-Derivative Test:

Find the matrix of second partials, $D^2f(x, y)$.

Evaluate this matrix at the point $a$. In other words, find $D^2f(a)$.

Then find the determinant. In other words, find $\det(D^2f(a))$, also denoted $|D^2f(a)|$. In other words, find the value at $a$ of the function $f_{xx}f_{yy} - f_{xy}^2$. Call this determinant $D$. (See note 1 below.)

Also take note of the value of $f_{xx}(a)$, which appears in the upper left corner of the matrix $D^2f(a)$.

(a) If $D > 0$ and $f_{xx}(a) > 0$, then $f$ has a (strict) local minimum at the critical point $a$.

(b) If $D > 0$ and $f_{xx}(a) < 0$, then $f$ has a (strict) local maximum at the critical point $a$.

(c) If $D < 0$, then $f$ has a saddle point at $a$.

(d) If $D = 0$, then the test is inconclusive. The function $f$ might have a local maximum at $a$, or it might have a local minimum, or it might have a saddle point. To decide which is the case, one must use some other kind of reasoning.

Note 1: $D$ may not be the best name for this, since $D$ often suggests derivative and this $D$ isn’t a derivative. So think $D$ for determinant or $D$ for discriminant, rather than $D$ for derivative. I’ve used “$D$” because that’s what the text uses here.

Note 2: The quantity $D$ is the same as what the text calls the discriminant of the Hessian, and “Hessian at $a$” is the name they give to the quantity $\frac{1}{2}hD^2f(a)h$, which is familiar from the second-order Taylor formula. This terminology is widely used in other contexts in mathematics, but I prefer, for this course, to refer to the determinant of the second-derivative matrix, rather than the discriminant of the Hessian.