Problem 9, §15.1, p709. For the function
\[ f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9 \]
find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

Solution. We have
\[ f_x(x, y) = 3x^2 - 3 = 3(x - 1)(x + 1) \]
\[ f_y(x, y) = 3y^2 - 12y = 3y(y - 4). \]

To find critical points, solve the system
\[ 3(x - 1)(x + 1) = 0 \]
\[ 3y(y - 4) = 0 \]
From the first equation, we must have \( x = 1 \) or \( x = -1 \). From the second equation, we must have \( y = 0 \) or \( y = 4 \). Thus there are four critical points, namely \((1,0)\), \((1,4)\), \((-1,0)\), and \((-1,4)\).

The matrix of second partial derivatives is
\[
\begin{bmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{bmatrix} = \begin{bmatrix}
  6x & 0 \\
  0 & 6y - 12
\end{bmatrix}
\]
Testing \((1,0)\). We have
\[ D = \text{det} \begin{bmatrix}
  6x & 0 \\
  0 & 6y - 12
\end{bmatrix}_{x=1, y=0} = \text{det} \begin{bmatrix}
  6 & 0 \\
  0 & -12
\end{bmatrix} = -72 < 0 \]
so \((1,0)\) gives a saddle point. The \(z\)-value at the saddle point is \( f(1,0) = 7 \).

Testing \((1,4)\). We have
\[ D = \text{det} \begin{bmatrix}
  6x & 0 \\
  0 & 6y - 12
\end{bmatrix}_{x=1, y=4} = \text{det} \begin{bmatrix}
  6 & 0 \\
  0 & 12
\end{bmatrix} = 72 > 0 \]
and
\[ f_{xx}(1,4) = 6 > 0 \]
so \((1,4)\) gives a local minimum. The local minimum value is \( f(1,4) = -25 \).

Testing \((-1,0)\). We have
\[ D = \text{det} \begin{bmatrix}
  6x & 0 \\
  0 & 6y - 12
\end{bmatrix}_{x=-1, y=0} = \text{det} \begin{bmatrix}
  -6 & 0 \\
  0 & -12
\end{bmatrix} = 72 > 0 \]
and
\[ f_{xx}(-1, 0) = -6 < 0 \]
so (-1, 0) gives a local maximum. The local maximum value is \( f(-1, 0) = 11 \).

Testing (-1, 4). This time
\[
D = \det \begin{bmatrix} 6x & 0 \\ 0 & 6y - 12 \end{bmatrix}_{x=-1, y=4} = \det \begin{bmatrix} -6 & 0 \\ 0 & 12 \end{bmatrix} = -72 < 0
\]
so (-1, 4) gives a saddle point. The z-value at the saddle point is \( f(-1, 4) = -23 \).

In summary, we have a local maximum of 11 at (-1, 0), a local minimum of -25 at (1, 4), and saddle points at \((x, y, z) = (1, 0, 7)\) and \((x, y, z) = (-1, 4, -23)\).

**Problem 13, §15.1, p709.** For the function
\[ f(x, y) = 8xy - \frac{1}{4}(x + y)^4 \]
find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

**Solution.** The partial derivatives are
\[
\begin{align*}
f_x(x, y) &= 8y - \frac{1}{4}(x + y)^3 \cdot 1 = 8y - (x + y)^3 \\
f_y(x, y) &= 8x - \frac{1}{4}(x + y)^3 \cdot 1 = 8x - (x + y)^3.
\end{align*}
\]
To find the critical points, solve the system
\[
\begin{align*}
8y - (x + y)^3 &= 0 \\
8x - (x + y)^3 &= 0
\end{align*}
\]
Subtracting the second equation from the first gives \( 8y - 8x = 0 \) or \( y = x \). Putting \( y = x \) into the first equation gives \( 8x - 8x^3 = 0 \) or \( 8x(1 - x)(1 + x) = 0 \). Thus we must have \( x = 0, x = 1, \) or \( x = -1 \) with \( y = x \). The critical points are therefore \((0, 0), (1, 1), \) and \((-1, -1)\).

The matrix of second partials is
\[
\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -3(x + y)^2 & 8 - 3(x + y)^2 \\ 8 - 3(x + y)^2 & -3(x + y)^2 \end{bmatrix}
\]
Testing (0, 0). We have
\[
D = \det \begin{bmatrix} -3(x + y)^2 & 8 - 3(x + y)^2 \\ 8 - 3(x + y)^2 & -3(x + y)^2 \end{bmatrix}_{x=0, y=0} = \det \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix} = -64 < 0
\]
so $(0,0)$ gives a saddle point. The corresponding $z$-value is 0.

Testing $(1,1)$. We have

$$D = \det \begin{bmatrix} -3(x + y)^2 & 8 - 3(x + y)^2 \\ 8 - 3(x + y)^2 & -3(x + y)^2 \end{bmatrix}_{x=1,y=1} = \det \begin{bmatrix} -12 & -4 \\ -4 & -12 \end{bmatrix} = 144 - 16 > 0$$

and

$$f_{xx}(1,1) = -12 < 0$$

so $(1,1)$ gives a local maximum. The local maximum value is $f(1,1) = 4$.

Testing $(-1,-1)$. We have

$$D = \det \begin{bmatrix} -3(x + y)^2 & 8 - 3(x + y)^2 \\ 8 - 3(x + y)^2 & -3(x + y)^2 \end{bmatrix}_{x=-1,y=-1} = \det \begin{bmatrix} -12 & -4 \\ -4 & -12 \end{bmatrix} > 0$$

and

$$f_{xx}(-1,-1) = -12 < 0.$$ 

Here, too, there is a local maximum value of 4. This might also have been predicted by the symmetry in the function, since $f(-x,-y) = f(x,y)$.

**Problem 20, §15.1, p709.** Suppose $f(x,y) = A - (x^2 + Bx + y^2 + Cy)$. What values of $A$, $B$, and $C$ give $f(x,y)$ a local maximum value of 15 at the point $(-2,1)$?

**Solution.** From the given information, we can conclude that $f(-2,1) = 15$, $f_x(-2,1) = 0$ and $f_y(-2,1) = 0$. Since $f(x,y) = A - (x^2 + Bx + y^2 + Cy)$, $f_x(x,y) = -2x - B$, and $f_y(x,y) = -2y - C$, these three pieces of information can be translated into the equations

$$A - (4 - 2B + 1 + C) = 15$$

$$4 - B = 0$$

$$-2 - C = 0$$

The last two equations tell us $B = 4$ and $C = -2$. Putting these values into the first equation and solving for $A$, we find $A - (-5) = 15$ or $A = 10$.

So far we know that the only possible values for $A$, $B$, and $C$ are $A = 10$, $B = 4$, and $C = -2$. We don’t know for sure, however, that these values actually work, since we’ve only arranged that $(-2,1)$ be a critical point yielding the right $z$-value. We need to check that the function $f(x,y) = 10 - (x^2 + 4x + y^2 - 2y)$ does have a local maximum at $(-2,1)$. We have $D(x,y) = 4$ and $f_{xx}(x,y) = -2$, so the second-derivative test assures us that the critical point $(-2,1)$ does yield a local maximum. Thus the values we seek are $A = 10$, $B = 4$, $C = -2$.

Note: If you prefer, you could confirm the local maximum by completing the square and using the
resulting formula: $f(x, y) = 15 - (x + 2)^2 - (y - 1)^2$. This function does have a maximum value of 15 at (-2,1).

**Problem 21, §15.1, p709.** At the point (1, 3), suppose that $f_x = f_y = 0$ and $f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} = 0$.

(a) What can you conclude about the behavior of the function near the point (1, 3)?

(b) Sketch a possible contour diagram.

**Solution.** (a) The information from the first partial derivatives tells us that the point (1,3) is a critical point for $f$. From the information about the second partials we find that $D(1, 3) = f_{xx}(1, 3)f_{yy}(1, 3) - [f_{xy}(1, 3)]^2 > 0$ with $f_{xx}(1, 3) > 0$. The second-derivative test therefore tells us that $f$ has a local minimum value at (1,3).

(b) There are lots of possibilities. Here’s one simple one. The levels, reading from outside in, should be decreasing. For instance, they might be 7, 6, 5, 4, 3, 2, 1. [If we knew $f_{xx} > 0$ everywhere and $f_{yy} > 0$ everywhere, then the outer contours should be closer together than the inner ones, but we don’t have that stronger information.]