Mathematics 205
HWK 15a Solutions
Section 15.1 p709

Problem 9, §15.1, p709. For the function

$$f(x,y) = x^3 + y^3 - 6y^2 - 3x + 9$$

find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

Solution. We have

$$f_x(x,y) = 3x^2 - 3 = 3(x-1)(x+1)$$
$$f_y(x,y) = 3y^2 - 12y = 3y\,(y-4).$$

To find critical points, solve the system

$$3(x-1)(x+1) = 0$$
$$3y\,(y-4) = 0$$

From the first equation, we must have $x = 1$ or $x = -1$. From the second equation, we must have $y = 0$ or $y = 4$. Thus there are four critical points, namely $(1,0)$, $(1,4)$, $(-1,0)$, and $(-1,4)$.

The matrix of second partial derivatives is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y-12 \end{bmatrix}$$

Testing $(1,0)$. We have

$$D = \det \begin{bmatrix} 6x & 0 \\ 0 & 6y-12 \end{bmatrix}_{x=1,y=0} = \det \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix} = -72 < 0$$

so $(1,0)$ gives a saddle point. The z-value at the saddle point is $f(1,0) = 7$.

Testing $(1,4)$. We have

$$D = \det \begin{bmatrix} 6x & 0 \\ 0 & 6y-12 \end{bmatrix}_{x=1,y=4} = \det \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix} = 72 > 0$$

and

$$f_{xx}(1,4) = 6 > 0$$

so $(1,4)$ gives a local minimum. The local minimum value is $f(1,4) = -25$.

Testing $(-1,0)$. We have

$$D = \det \begin{bmatrix} 6x & 0 \\ 0 & 6y-12 \end{bmatrix}_{x=-1,y=0} = \det \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix} = 72 > 0$$

and

$$f_{xx}(-1,0) = -6 < 0$$

so $(-1,0)$ gives a local maximum. The local maximum value is $f(-1,0) = 11$.

Testing $(-1,4)$. This time

$$D = \det \begin{bmatrix} 6x & 0 \\ 0 & 6y-12 \end{bmatrix}_{x=-1,y=4} = \det \begin{bmatrix} -6 & 0 \\ 0 & 12 \end{bmatrix} = -72 < 0$$

so $(-1,4)$ gives a saddle point. The z-value at the saddle point is $f(-1,4) = -23$.

In summary, we have a local maximum of 11 at $(-1,0)$, a local minimum of -25 at $(1,4)$, and saddle points at $(x,y,z) = (1,0,7)$ and $(x,y,z) = (-1,4,-23)$.

Problem 13, §15.1, p709. For the function

$$f(x,y) = 8xy - \frac{1}{4}(x+y)^4$$

find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

Solution. The partial derivatives are

$$f_x(x,y) = 8y - \frac{1}{4}4(x+y)^3 \cdot 1 = 8y - (x+y)^3$$

$$f_y(x,y) = 8x - \frac{1}{4}4(x+y)^3 \cdot 1 = 8x - (x+y)^3.$$

To find the critical points, solve the system

$$8y - (x+y)^3 = 0$$

$$8x - (x+y)^3 = 0$$

Subtracting the second equation from the first gives $8y - 8x = 0$ or $y = x$. Putting $y = x$ into the first equation gives $8x - 8x^3 = 0$ or $8x(1-x)(1+x) = 0$. Thus we must have $x = 0, x = 1$, or $x = -1$ with $y = x$. The critical points are therefore $(0,0)$, $(1,1)$, and $(-1,-1)$.

The matrix of second partials is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -3(x+y)^2 & 8-3(x+y)^2 \\ 8-3(x+y)^2 & -3(x+y)^2 \end{bmatrix}$$

Testing $(0,0)$. We have

$$D = \det \begin{bmatrix} -3(x+y)^2 & 8-3(x+y)^2 \\ 8-3(x+y)^2 & -3(x+y)^2 \end{bmatrix}_{x=0,y=0} = \det \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix} = -64 < 0$$

so $(0,0)$ gives a saddle point. The corresponding z-value is 0.

Testing $(1,1)$. We have

$$D = \det \begin{bmatrix} -3(x+y)^2 & 8-3(x+y)^2 \\ 8-3(x+y)^2 & -3(x+y)^2 \end{bmatrix}_{x=1,y=1} = \det \begin{bmatrix} -12 & -4 \\ -4 & -12 \end{bmatrix} = 144-16 > 0$$

and

$$f_{xx}(1,1) = -12 < 0$$

so $(1,1)$ gives a local maximum. The local maximum value is $f(1,1) = 4$.

Testing $(-1,-1)$. We have

$$D = \det \begin{bmatrix} -3(x+y)^2 & 8-3(x+y)^2 \\ 8-3(x+y)^2 & -3(x+y)^2 \end{bmatrix}_{x=-1,y=-1} = \det \begin{bmatrix} -12 & -4 \\ -12 & --4 \end{bmatrix} > 0$$

and

$$f_{xx}(-1,-1) = -12 < 0.$$

Here, too, there is a local maximum value of 4. This might also have been predicted by the symmetry in the function, since $f(-x,-y) = f(x,y)$.

Problem 20, §15.1, p709. Suppose $f(x,y) = A - (x^2 + Bx + y^2 + Cy)$. What values of A, B, and C give $f(x,y)$ a local maximum value of 15 at the point $(-2,1)$?

Solution. From the given information, we can conclude that $f(-2,1) = 15$, $f_x(-2,1) = 0$ and $f_y(-2,1) = 0$. Since $f(x,y) = A-(x^2+Bx+y^2+Cy)$, $f_x(x,y) = -2x-B$, and $f_y(x,y) = -2y-C$, these three pieces of information can be translated into the equations

$$A - (4 - 2B + 1 + C) = 15$$

$$4 - B = 0$$

$$-2 - C = 0$$

The last two equations tell us $B = 4$ and $C = -2$. Putting these values into the first equation and solving for A, we find $A - (-5) = 15$ or $A = 10$.

So far we know that the only possible values for A, B, and C are $A = 10$, $B = 4$, and $C = -2$. We don't know for sure, however, that these values actually work, since we've only arranged that $(-2,1)$ be a critical point yielding the right z-value. We need to check that the function $f(x,y) = 10-(x^2+4x+y^2-2y)$ does have a local maximum at $(-2,1)$. We have $D(x,y) = 4$ and $f_{xx}(x,y) = -2$, so the second-derivative test assures us that the critical point $(-2,1)$ does yield a local maximum. Thus the values we seek are $A = 10, B = 4, C = -2$.

Note: If you prefer, you could confirm the local maximum by completing the square and using the resulting formula: $f(x,y) = 15 - (x+2)^2 - (y-1)^2$. This function does have a maximum value of 15 at (-2,1).

Problem 21, §15.1, p709. At the point $(1,3)$, suppose that $f_x = f_y = 0$ and $f_{xx} > 0,\ f_{yy} > 0,\ f_{xy} = 0$.
(a) What can you conclude about the behavior of the function near the point $(1,3)$?
(b) Sketch a possible contour diagram.

Solution. (a) The information from the first partial derivatives tells us that the point $(1,3)$ is a critical point for f. From the information about the second partials we find that $D(1,3) = f_{xx}(1,3)f_{yy}(1,3) - [f_{xy}(1,3)]^2 > 0$ with $f_{xx}(1,3) > 0$. The second-derivative test therefore tells us that f has a local minimum value at $(1,3)$.

(b) There are lots of possibilities. Here's one simple one. The levels, reading from outside in, should be decreasing. For instance, they might be 7, 6, 5, 4, 3, 2, 1. [If we knew $f_{xx} > 0$ everywhere and $f_{yy} > 0$ everywhere, then the outer contours should be closer together than the inner ones, but we don't have that stronger information.]