Mathematics 205
HWK 15b Solutions
Section 15.2 p716

Problem 3, §15.2, p716. Does the function $f(x, y) = -2x^2 - 7y^2$ have global maxima and minima? Explain.

Solution. The surface $z = -2x^2 - 7y^2$ is a paraboloid that opens downward. It has a global maximum at $(0, 0)$ and no local or global minima.

Or one could argue that the level curves are ellipses that surround the origin, with the z-levels getting lower as the ellipses get bigger. This confirms the global maximum at $(0, 0)$.

Yet another line of reasoning would be that $f(0, 0) = 0$ while $f(x, y) < 0$ for every (x, y) except $(0, 0)$. As before, conclude there's a global maximum at $(0, 0)$. Then to show there's no global minimum, note, for instance, that $f(x, 0) \longrightarrow -\infty$ as $x \longrightarrow \pm\infty$. Note that this reasoning alone wouldn't preclude the possibility of other maxima or of local minima.

Problem 5, §15.2, p716. Find the global maximum and minimum of the function $z = x^2 + y^2$ on the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and say whether it occurs on the boundary of the square. [Hint: Use graphs.]

Solution. The contours for $z = x^2 + y^2$ are circles centered at the origin, with larger circles corresponding to higher z-values. Draw yourself a picture.

To identify the global maximum we'll look for the points on the square that are farthest from the origin and therefore lie on the largest possible contour circle. Thus for (x, y) restricted to the specified square, the highest z-value will occur on the contour circle that goes through the four corners of the square. This is the contour $x^2 + y^2 = 1^2 + 1^2 = 2$. So the specified global maximum value is $z = 2$, occurring at all four of the points $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.

For (x, y) restricted to the specified square, the lowest z-value will occur at the origin, which is the center of the concentric contour circles. The global minimum value is $z = 0$.

One could also argue from the graph of the surface, but that reasoning is more complicated to describe.

Problem 24, §15.2, p716. An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length and height less than or equal to 135 cm. Find the dimensions of the suitcase of maximum volume that a passenger may carry under this regulation.

Solution. Denote the width, length, and height by W, L, and H, respectively (with each measured in centimeters). We may as well assume that $W + L + H = 135$. Otherwise one could increase one or more of the dimensions and get a larger volume. We with to maximize the volume

$V = LWH$. Using the condition $W + L + H = 135$, we can reformulate the problem as: find the values for $W > 0$ and $H > 0$ that will maximize $V = HW(135-W-H) = 135HW - HW^2 - H^2W$. Since

$$\frac{\partial V}{\partial W} = 135H - 2HW - H^2$$

and

$$\frac{\partial V}{\partial H} = 135W - W^2 - 2HW$$

we find critical points by solving the system

$$135H - 2HW - H^2 = 0$$

$$135W - W^2 - 2HW = 0$$

Subtracting the second equation from the first we find

$$135H - H^2 - 135W + W^2 = 0$$

which can be factored as

$$(H - W)(135 - H - W) = 0$$

This gives us $H = W$ or $H + W = 135$. We cannot have $H + W = 135$, lest $L = 0$ and we have zero volume. So we must have $H = W$. Putting $H = W$ into the equation

$$135H - 2HW - H^2 = 0$$

we find

$$135H - 2H^2 - H^2 = 0$$

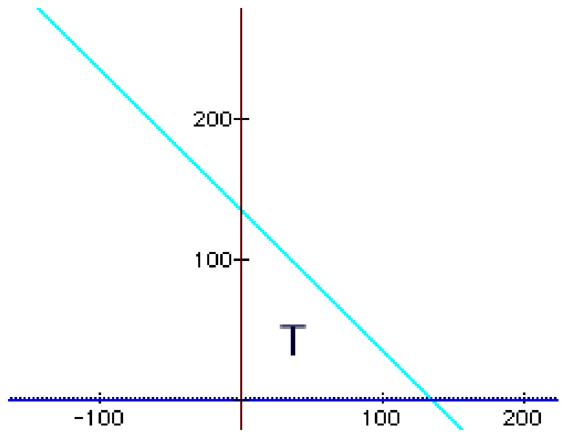
or

$$H(135 - 3H) = 0.$$

Since H cannot be 0, we have $3H = 135$, so $H = 45$. Then $W = H$ gives $W = 45$. Finally, $L + W + H = 135$, with $H = W = 45$ gives $L = 45$. The three dimensions of the suitcase should each be 45 centimeters. [Of course, no one actually makes suitcases with all three dimensions equal. If you've ever tried to carry a wide suitcase by the handle you know why.]

If we wanted to confirm that these dimensions give a local maximum, we could use the second-derivative test. However, since this function is not a quadratic, having a single local maximum won't necessarily guarantee a global maximum. So some other reasoning will be needed to confirm that we have a global maximum. In practice one might argue from the physical considerations in the problem.

To rigorously confirm, however, that we have a global maximum we could argue as follows. Consider the set of points (H, W) where $H \geq 0$, $W \geq 0$ and $H + W \leq 135$. This triangular region, T, say is a closed, bounded set that includes all of the points (H, W) that are feasible for the suitcase problem.



The volume function $V = HW(135 - W - H)$ is continuous on the triangular region T, so it has a global maximum value on T. The global maximum can only occur at an interior critical point or at a boundary point of T. But at each point of the boundary we will have $V = 0$ (since we'll have $H = 0$ or $W = 0$ or $H + W = 135$). Since the volume value at the sole interior critical point is higher than the volume value at all the boundary points, the global maximum must occur at the critical point. In other words, using $H = W = L = 45$ gives a global maximum, not just a local maximum.

Problem 27, §15.2, p716. Let $f(x, y) = x^2(y + 1)^3 + y^2$. Show that f has only one critical point, namely $(0, 0)$, and that point is a local minimum but not a global minimum. Contrast this with the case of a function with a single local minimum in one-variable calculus. [Also see the assignment page for HWK 16.]

Solution. We have

$$f_x(x, y) = 2x(y + 1)^3$$
$$f_y(x, y) = 3x^2(y + 1)^2 + 2y$$

For $f_x = 0$ we must have $x = 0$ or $y = -1$. However, $y = -1$ would make $f_y \neq 0$. Therefore we must have $x = 0$. Using $x = 0$ in $f_y = 0$ tells us that $y = 0$, as well. So the only critical point is, in fact, $(0, 0)$.

The matrix of second partials is

$$\begin{bmatrix} 2(y+1)^3 & 6x(y+1)^2 \\ 6x(y+1)^2 & 6x^2(y+1) + 2 \end{bmatrix}.$$

This gives

$$D = \det \begin{bmatrix} 2(y+1)^3 & 6x(y+1)^2 \\ 6x(y+1)^2 & 6x^2(y+1)+2 \end{bmatrix}_{x=0,y=0} = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$$

and

$$f_{xx}(0,0) = 2 > 0.$$

According to the second-derivative test, $(0,0)$ gives a local minimum for f.

We can show that $(0,0)$ does not give a global minimum by showing that there is no global minimum. Note that $f(0,0) = 0$. Note that $(y+1)^3$ dominates y^2 in absolute value when $|y|$ is large and that $(y+1)^3 < 0$ when y is negative with $|y|$ large. Consider the section $g(y) = f(1,y) = (y+1)^3 + y^2$. As $y \longrightarrow -\infty$, we will have $g(y) \longrightarrow -\infty$. Thus $f(1,y)$ takes on arbitrarily large negative values, so $f(x,y)$ does as well. This shows that f cannot have a global minimum value. For if it did, then $f(x,y)$ could not go below that value, yet $f(x,y)$ goes below every possible value.

The function in this problem has a single critical point, which yields a local minimum, yet it does not have a global minimum. If this were a function of one variable only, such a situation would not be possible. For functions of one variable, a single critical point yielding a local minimum automatically yields a global minimum. Intuitively, to avoid having a global minimum, the function would have to change directions at some point and go back down again. This can't happen without introducing a second critical point. With the extra dimension there's more room to maneuver. You might imagine having, in 3-space, a surface $z = f(x,y)$ with no critical points and extending to both arbitrarily high z-values and arbitrarily low z-values. Then pretend the surface is made out of stretchable rubber and poke your finger into it just enough to create a local minimum but no other critical points. That's rather what the surface in this problem is like.

Note, however: the algebra of quadratic surfaces assures us that a phenomenon like this can't happen for them. For quadratic surfaces with a single local minimum [maximum], the local minimum [maximum] is a global minimum [maximum].