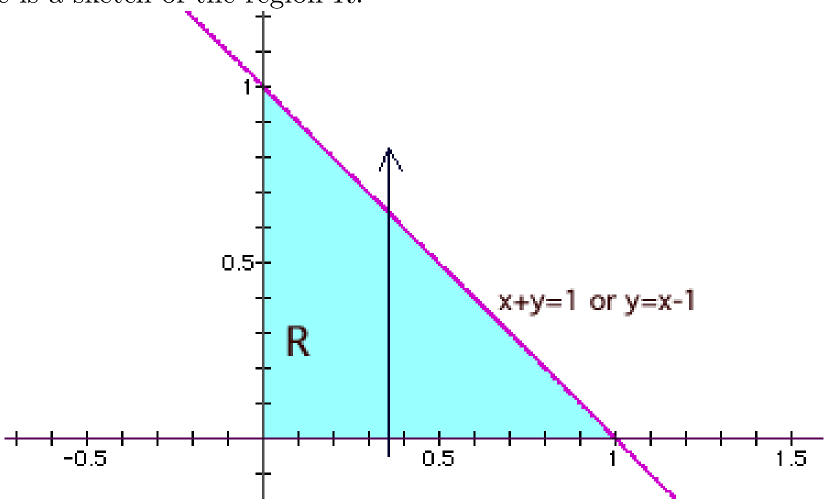
Mathematics 205
HWK 19b Solutions
Section 16.2 p750

Problem 17, §16.2, p750. Evaluate $\int_R xy\,dA$, where R is the triangular region for which $x+y \le 1$, $x \ge 0$, $y \ge 0$.

Solution. Here is a sketch of the region R.



There's a lot of symmetry here, and it really won't matter which order we integrate in. Note that the equation for the top, slanted edge of the region R is $x+y=1$. If we shoot vertical arrows (so first fix x and integrate with respect to y), a vertical arrow for fixed x will enter R when $y=0$ and leave R when $y=1-x$. Then x will have to vary over the interval from $x=0$ to $x=1$. This gives us

$$\int_R xy\,dA = \int_0^1 x\left(\int_0^{1-x} y\,dy\right)dx = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{1}{2}x(1-x)^2\,dx.$$

To complete the integration, one method is to expand the polynomial. Writing $x(1-x)^2 = x - 2x^2 + x^3$ we have

$$\int_0^1 \frac{1}{2}x(1-x)^2\,dx = \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)dx = \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{1}{24} \approx 0.0416667$$

Combining these results, we have

$$\int_R xy\,dA = \frac{1}{24} \approx 0.0416667.$$

Alternatively, one can do the final integration using integration by parts. With $u=x$ and $dv=(1-x)^2$, so that $du=dx$ and $v = -\frac{(1-x)^3}{3}$, we have

$$\int x(1-x)^2\,dx = x(1-x)^2 + \int \frac{(1-x)^3}{3}\,dx = x(1-x)^2 - \frac{(1-x)^4}{12}.$$

This gives us

$$\int_R xy\,dA = \int_0^1 \frac{1}{2}x(1-x)^2\,dx = \left[\frac{1}{2}x(1-x)^2 - \frac{1}{24}(1-x)^4\right]_0^1 = \frac{1}{24}$$

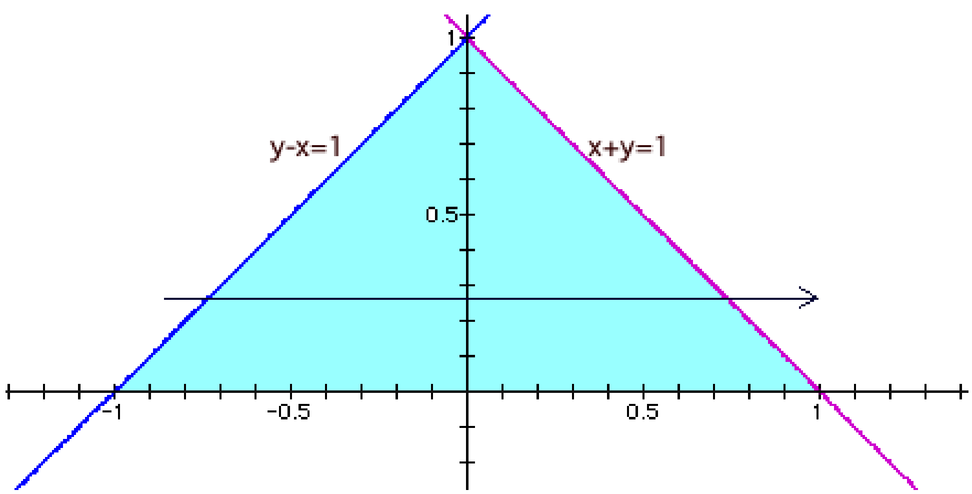
the same result as obtained above.

If you choose to shoot horizontal arrows, integrating first with respect to x and then with respect to y, the calculations turn out to be identically the same, but with x and y reversed. (This is because the integrand doesn't change when x and y are switched, and nor does the description of the region of integration.) To be specific, you'll have

$$\int_R xy\,dA = \int_0^1 y\left(\int_0^{1-y} x\,dx\right)dy = \cdots = \frac{1}{24}.$$

Problem 21, §16.2, p750. Evaluate $\int_R (2x+3y)^2\,dA$, where R is the triangular region with vertices $(-1,0)$, $(0,1)$, and $(1,0)$.

Solution. Here's a sketch of the region showing the equations for the top two sides.



As the sketch suggests, it will work best to shoot horizontal arrows, integrating first with respect to x. The alternative would require us to set up two separate double iterated integrals, compute them, and add to get the integral we want. Shooting a horizontal arrow for fixed y, the arrow will enter R when $x = y - 1$ (from the equation on the top left) and leave when $x = 1 - y$ (from the equation on the top right). Then we need to have y vary from 0 to 1. This gives us

$$\int_R (2x+3y)^2\,dA = \int_0^1 \int_{y-1}^{1-y} (2x+3y)^2\,dx\,dy.$$

Again, we have a couple of options for performing the remaining calculation. One is to expand the integrand and integrate term by term. Better yet, expand and notice that the integral of the middle term will vanish (i.e. will have the value 0) because of the symmetry in the integrand and the region of integration). Using this method we have

$$\begin{aligned}
\int_R (2x+3y)^2\,dA &= \int_R (4x^2+12xy+9y^2)\,dA \\
&= \int_R 4x^2\,dA + \int_R 12xy\,dA + \int_R 9y^2\,dA \\
&= \int_R 4x^2\,dA + 0 + \int_R 9y^2\,dA \\
&= \int_0^1 \int_{y-1}^{1-y} (4x^2+9y^2)\,dx\,dy \\
&= \int_0^1 \left[\frac{4x^3}{3} + 9xy^2\right]_{x=y-1}^{x=1-y} dy \\
&= \int_0^1 \left(\frac{4(1-y)^3}{3} - \frac{4(y-1)^3}{3} + 18y^2 - 18y^3\right) dy \\
&= \left[-\frac{(1-y)^4}{3} - \frac{(y-1)^4}{3} + 6y^3 - \frac{9y^4}{2}\right]_0^1 \\
&= 0 - 0 + 6 - \frac{9}{2} - [-\frac{1}{3} - \frac{1}{3} + 0 - 0] \\
&= 6 - \frac{9}{2} + \frac{2}{3} = \frac{13}{6}.
\end{aligned}$$

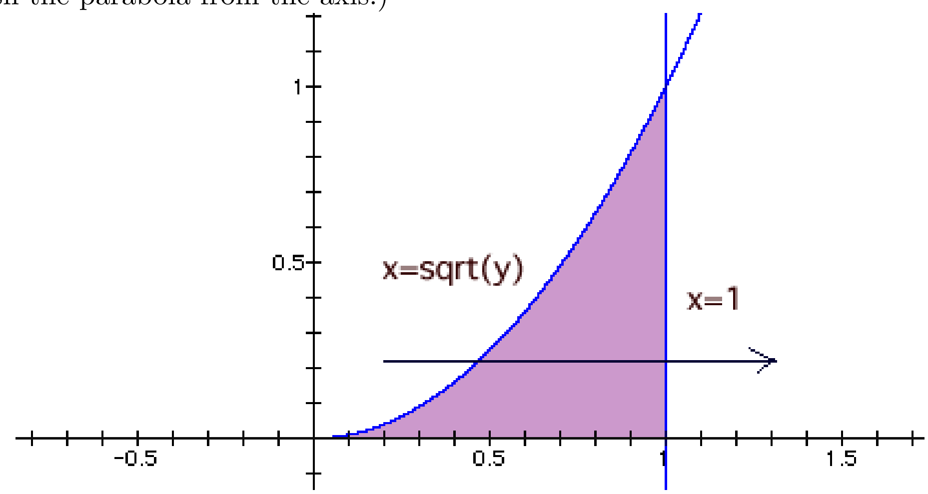
Another option would be to leave the integrand as is and use tiny substitutions to find the antiderivatives we need. In other words, we could compute as follows:

$$\begin{aligned}
\int_R (2x+3y)^2\,dA &= \int_0^1 \int_{y-1}^{1-y} (2x+3y)^2\,dx\,dy = \int_0^1 \left[\frac{(2x+3y)^3}{2\cdot 3}\right]_{x=y-1}^{x=1-y} dy \\
&= \frac{1}{6}\int_0^1 \left[(2+y)^3 - (5y-2)^3\right]\,dy \\
&= \frac{1}{6}\left[\frac{1}{4}(2+y)^4 - \frac{1}{5\cdot 4}(5y-2)^4\right]_0^1 \\
&= \frac{1}{24}\left[81 - \frac{81}{5} - 16 - \frac{16}{5}\right] \\
&= \frac{1}{24}\frac{4}{5}(81-16) \\
&= \frac{65}{30} \\
&= \frac{13}{6}
\end{aligned}$$

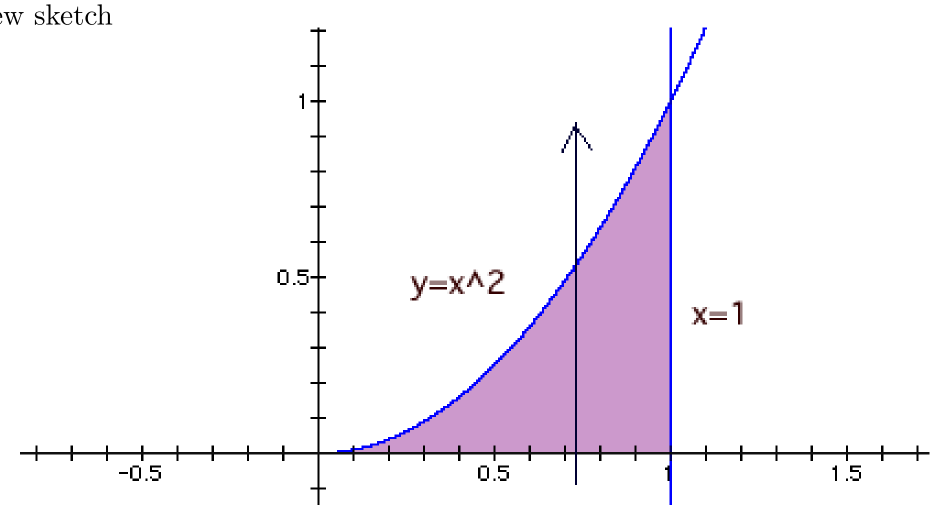
as above.

Problem 29, §16.2, p750. Evaluate the integral $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\,dx\,dy$ by reversing the order of integration.

Solution. Begin by sketching the region of integration. For the iterated integral we are given, the inner integral tells us that for fixed y, a horizontal arrow (shot in the direction of increasing x) will enter the region when $x = \sqrt{y}$ and leave the region when $x = 1$. Thus the left side of the region has the equation $x = \sqrt{y}$ as its boundary curve, and the right side is the line $x = 1$. Sketch this much. Notice that $x = \sqrt{y}$ meets $x = 1$ where $(x, y) = (1, 1)$, and the top limit on the outer integral is given as $y = 1$, so the intersection point of the parabola and the line $x = 1$ is the top point of the region. Since the outer integral has $y = 0$ as the lower limit of integration, the line $y = 0$, i.e. the x-axis, forms the lower edge of the region. Thus the sketch (showing a typical horizontal arrow) for the iterated integral we are given would look like this. (Note: the region does extend to $(0, 0)$ on the left. The sketch doesn't quite show that, because the scale makes it hard to distinguish the parabola from the axis.)



To reverse the order of integration, imagine shooting a vertical arrow. For fixed x, the corresponding vertical arrow enters the region when $y = 0$ and leaves it when it hits the parabola, so when $y = x^2$. Thus the new sketch



gives us the reversed iterated integral

$$\int_0^1 \int_0^{x^2} \sqrt{2+x^3}\, dy\, dx.$$

Now evaluate:

$$\begin{aligned}\int_0^1 \int_0^{x^2} \sqrt{2+x^3}\, dy\, dx &= \int_0^1 x^2\sqrt{2+x^3}\, dx \\ &= \frac{2}{3}\frac{(2+x^3)^{\frac{3}{2}}}{3\cdot 3} \\ &= \frac{2\cdot 3\sqrt{3}}{9} - \frac{2\cdot 2\sqrt{2}}{9} \\ &= \frac{2}{9}(3\sqrt{3}-2\sqrt{2}).\end{aligned}$$

Problem 45, §16.2, p750. The function $f(x,y) = ax + by$ has an average value of 20 on the rectangle $0 \le x \le 2,\ \ 0 \le y \le 3$.
(a) What can you say about the constants a and b?
(b) Find two different choices for f that have average value 20 on the rectangle, and give their contour diagrams on the rectangle

Solution. (a) We know that the average value for $f(x,y)$ over some region R in the xy-plane is the ratio

$$\frac{\displaystyle\int_R f(x,y)\, dA}{\displaystyle\int_R 1\cdot\, dA}.$$

The numerator represents the area of R, which in this case is 6. Evaluate the numerator, with $f(x,y) = ax + by$ and R the specified rectangle:

$$\begin{aligned}\int_R f(x,y)\, dA &= \int_0^3 \int_0^2 (ax+by)\, dx\, dy \\ &= \int_0^3 \left[\frac{1}{2}ax^2 + bxy\right]_{x=0}^{x=2} dy \\ &= \int_0^3 (2a+2by)\, dy \\ &= \left[2ay + by^2\right]_{y=0}^{y=3} \\ &= 6a + 9b.\end{aligned}$$

Thus the average value in question is $\frac{1}{6}(6a+9b)$, which we are told must be 20. So we must have $6a + 9b = 120$ or $2a + 3b = 40$. This is as much as we can say about a and b.

(b) All we have to do to find two different choices for f that will have the right average value is to pick two pairs of values for a and b that make $2a + 3b = 40$. For instance, we could use $a = 8$ and $b = 8$; or we could use $a = 2$ and $b = 18$. There are lots of other possiblities. I'll leave it to you to do contour diagrams. The contours will be straight lines, of course, equally spaced for equal increments in the f-value.