Mathematics 205
HWK 19c Solutions
Section 16.3 p755

Problem 17, §16.3, p755. Find the mass of the solid bounded by the xy-plane, the yz-plane, the xz-plane, and the plane

$$\frac{x}{3}+\frac{y}{2}+\frac{z}{6}=1$$

if the density is given by the function

$$\delta(x,y,z)=x+y.$$

Solution. To be added later. We set this one up in class. What's left is the (messy but manageable) integration.

Problem 18, §16.3, p755. Find the average value of the sum of the squares of three numbers x, y, z, where each number is between 0 and 2.

Solution. Let $f(x,y,z)=x^2+y^2+z^2$, and let R denote the brick in 3-space that is given by the conditions $0\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 2$. Then the number we want is the average value of f over R. It is easy to compute that the volume of R is 8. We need to find $\int_R f\,dV$. We can use an iterated integral, in any one of the 6 possible orders of integration. For instance,

$$\begin{aligned}
\int_R f(x,y,z)\,dV &= \int_0^2\int_0^2\int_0^2 (x^2+y^2+z^2)\,dx\,dy\,dz\\
&= \int_0^2\int_0^2 \left[\frac{x^3}{3}+xy^2+xz^2\right]_{x=0}^{x=2} dy\,dz\\
&= \int_0^2\int_0^2 \left(\frac{8}{3}+2y^2+2z^2\right) dy\,dz\\
&= \int_0^2 \left[\frac{8y}{3}+\frac{2y^3}{3}+2yz^2\right]_{y=0}^{y=2} dz\\
&= \int_0^2 \left(\frac{32}{3}+4z^2\right) dz\\
&= \left[\frac{32z}{3}+\frac{4z^3}{3}\right]_{z=0}^{z=2}\\
&= 32
\end{aligned}$$

Now divide by the volume of R to find that the specified average value is

$$\frac{\int_R f\,dV}{\text{volume of } R}=\frac{32}{8}=4.$$

I don't know about you, but I found this result a little surprising – surprising, that is, that it came out to be a whole number. I was expecting some weird fraction.

Problem 21, §16.3, p755. Let W be the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 2$. Decide (without calculating) whether $\int_W x\,dV$ is positive, negative, or zero.

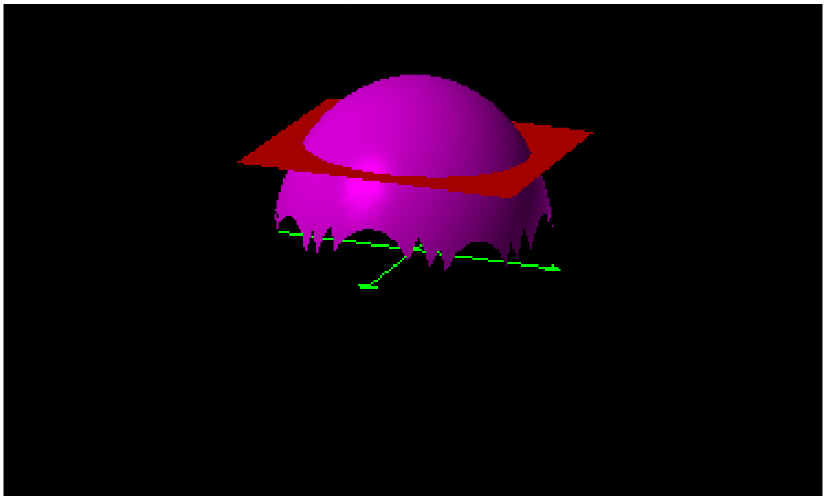
Solution. This cone has its tip at the origin, and it expands outward and upward until $z = 2$. The cross sections parallel to the xy-plane are all disks with centers on the z-axis. On the half of the cone that lies in front of the yz-plane, the integrand, x, will be positive. On the other half, which lies behind the yz-plane, the integrand will be negative. Moreover each point in the front half has a matching point in the back half where the integrand takes exactly the opposite value. Therefore the integral over the front half and the integral over the back half will cancel each other out. The total integral will be zero. A shorter description of the key facts would be: the integrand is an odd function of the variable x and the region is symmetric with respect to the plane $x = 0$. This symmetry makes the total integral zero.

Problem 30, §16.3, p755. Let W be the solid half-cone bounded by $z = \sqrt{x^2 + y^2}$, $z = 2$, and the yz-plane with $x \geq 0$. Decide whether the integral $\int_W x\,dV$ is positive, negative, or zero.

Solution. Every point of the region of integration satisfies $x \geq 0$, and x is the function we are integrating. Thus the integrand is ≥ 0 (and it only equals zero on the leftmost face of the region), so the integral is positive.

Problem 37, §16.3, p755. The figure in the text shows part of a spherical ball of radius 5 cm, namely the part that goes from the bottom of the sphere up 2 cm (so the part that lies between a plane just tangent to the bottom of the sphere and a parallel plane 2 cm higher). Write an iterated triple integral that represents the volume of this region.

Solution. Call the solid region W. Then we want to write $\int_W 1 \cdot dV$ as an iterated triple integral. First we need to introduce a coordinate system. It will probably be simplest to turn W upside down first and to use a coordinate system that puts the top of the sphere at the point $(0, 0, 5)$ so the center is at the origin. Then the sphere has the equation $x^2 + y^2 + z^2 = 25$, and the solid W is that portion of the ball $x^2 + y^2 + z^2 \leq 25$ that lies above the plane $z = 3$. (See sketch on the next page.)



It will be easiest to integrate first with respect to z. Fixing (x, y) and shooting a z-arrow upward, we see that the arrow enters the solid where $z = 3$ and leaves it where z hits the sphere, so where $z = \sqrt{25 - x^2 - y^2}$. Then the point (x, y) must vary over the disk in the (x, y)-plane that is the shadow of the disk formed by the intersection of the sphere $x^2 + y^2 + z^2 = 25$ with the plane $z = 3$. This intersection circle has equation $x^2 + y^2 + 9 = 25$ or $x^2 + y^2 = 16$. In other words, we need (x, y) to vary over the disk in the xy-plane that lies inside a circle of radius 4 centered at the origin. This gives us either

$$\text{volume of } W = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} dz\, dy\, dx$$

or

$$\text{volume of } W = \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{3}^{\sqrt{25-x^2-y^2}} dz\, dx\, dy.$$

Better yet, take advantage of the symmetry and use, say,

$$\text{volume of } W = 4 \int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} dz\, dy\, dx$$

Of course this integral is still messy to calculate using rectangular coordinates. You might like to think about how you could find this volume more easily.