Mathematics 205
HWK 21 Solutions
Section 16.5 p766

Problem 5, §16.5, p766. For the region shown (a rectangular slab of dimensions $1 \times 3 \times 5$; see the text), choose coordinates and set up a triple integral, including limits of integration, for a density function f over the given region.

Solution. Note that we're setting up an integral whose value is the mass of the given rectangular slab (with given but unspecified density function f). Use rectangular (or Cartesian) coordinates. I'll introduce my coordinate system so that the rectangular slab is specified by the conditions $0 \le x \le 1,\ 0 \le y \le 3,\ 0 \le z \le 5$. (In other words, put the origin at the lower left back corner, with the x-axis pointing in the direction where the slab has dimension 1, the y-axis the slab direction of dimension 3, and the z-axis in the slab direction of dimension 5. I'll write $f(P)$ for the density at point P, where the f-values are understood to be expressed in rectangular coordinates using the chosen coordinate system. For dV use the rectangular equivalent $dx\,dy\,dz$ (in some order or other). The limits of integration will come from the inequalities listed earlier. We can set up the integral for any one of the six possible orders (although the specific density function, were it given, might dictate a preference for one order over another). It is important, however, to match up the limits of integration in the iterated integral with the correct choice from dx, dy, dz. One possibility would be

$$\int_0^5 \int_0^3 \int_0^1 f(P)\,dz\,dy\,dx.$$

Many other correct answers are possible, especially since the given f is unspecified and the choice of coordinate system is up to you!

Problem 7, §16.5, p766. For the region shown (a ninety-degree wedge cut from a solid right circular cylinder of radius 2 and height 4), choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

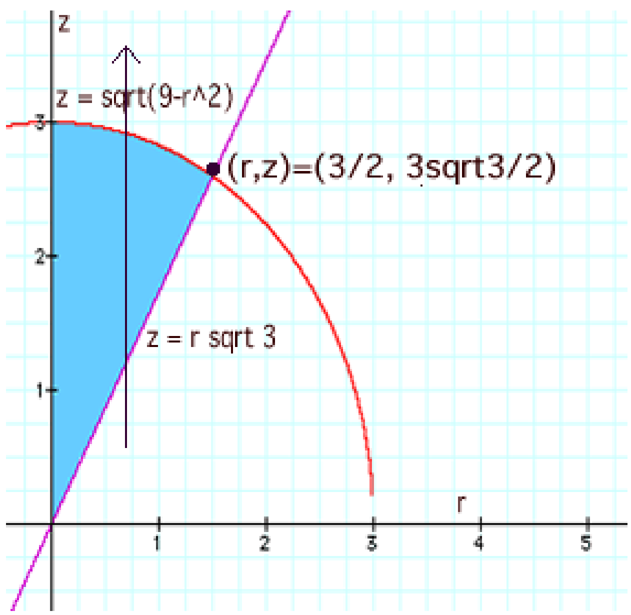
Solution. We could think of this region as being obtained from a wheel of cheese, of radius 2 and height 4, by cutting the cheese in half vertically, and then cutting each half in half again, vertically. The two cuts would cut the wheel of cheese into four pieces. Take one such piece, and that's our solid for this problem. This region calls for cylindrical coordinates. I'll introduce the coordinate system so that the z-axis lies along the "spine" of the wedge, with the origin at the bottom, one boundary face where $\theta = 0$ and one boundary face where $\theta = \frac{\pi}{2}$. I'll write $f(P)$ for the density at point P, expressed in polar coordinates according to the chosen coordinate system. For dV we must use the cylindrical equivalent $r\,dr\,dz\,d\theta$ (in some order or other). Limits of integration are given by the specifications $0 \le z \le 4,\ 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}$. One possibility for the desired integral would be

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^4 f(P) r\,dz\,dr\,d\theta.$$

Here, too, many other correct answers are possible. Do be sure, though, that your limits of integration match your $d*$.

Problem 9, §16.5, p766. For the region shown (an ice-cream-cone-shaped piece of a sphere, cut from a sphere of radius 3 and having a central angle of $\frac{\pi}{3}$), choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

Solution. Let's put the origin at the tip of the sphere, with the z-axis as the central axis of the ice-cream cone solid. Cylindrical coordinates will be better than rectangular. (One could also use spherical coordinates, probably more easily than cylindrical, but we'll leave that for later.) I'll write $f(P)$ for the density at point P of the solid, with density expressed in cylindrical coordinates. Use $dV = r\,dr\,dz\,d\theta$ (in some order). To determine limits of integration, notice that the cross sections for fixed θ all have the same shape: a sort of triangle with one side on the z-axis, one side on the outside surface of the cone, and one side on the spherical top of the ice-cream cone. (See sketch below.) The angle at the bottom tip is $\frac{\pi}{6}$. Using a little trigonometry we can write the equation of the lower right side as $z = r\sqrt{3}$. The equation of the spherical side is $z = \sqrt{9 - r^2}$. The point of intersection of the spherical side with the conical side occurs where $r\sqrt{3} = \sqrt{9 - r^2}$, hence where $(r, z) = (\frac{3}{2}, \frac{3\sqrt{3}}{2})$. Shooting a vertical z-arrow as shown, see that the arrow enters the cross section where $z = r\sqrt{3}$ and leaves where $z = \sqrt{9 - r^2}$. The r-values must vary from $r = 0$ to $r = \frac{3\sqrt{3}}{2}$.



This gives us the integral

$$\int_0^{2\pi} \int_0^{\frac{3}{2}} \int_{r\sqrt{3}}^{\sqrt{9-r^2}} f(P)\,r\,dz\,dr\,d\theta.$$

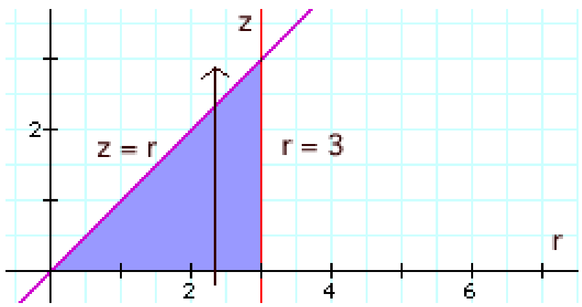
Other correct answers are possible.

Problem 13, §16.5, p766. (a) Convert the following triple integral to spherical coordinates:

$$\int_0^{2\pi} \int_0^3 \int_0^r r\,dz\,dr\,d\theta.$$

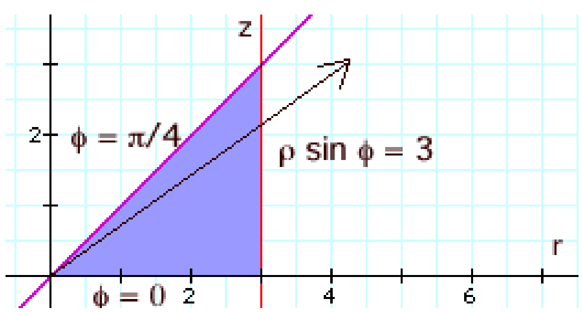
(b) Evaluate either the original integral or your answer to part (a).

Solution. Since the outermost integral is taken with respect to θ, the inner two integrals give an integral over the cross section for fixed θ. So let's try to identify the sketch, in a zr-halfplane, for that cross section. From the innermost integral, we see that, for fixed r, a z-arrow would enter this cross section where $z = 0$ and leave it where $z = r$. Then r would vary from 0 to 3. This gives cross sections like this (they're all the same, no matter what θ is):



The limits on the outermost integral tells us that this cross section must be rotated through a full revolution in order to generate the full solid. Thus the solid we're working with looks a little like a cylindrical pencil sharpener (except that it doesn't actually have a hole to put the pencil in!). Or is could be like a steeply sloped football stadium except that there's no room in the middle for the football field. Putting it in more technical language, it is the solid between the cone $z = r$, the xy-plane, and the cylinder $r = 3$. In any case, we won't actually need to visualize the whole solid, because we have a clear picture of what the cross sections for fixed θ look like. We can use these cross sections to make our conversion.

In fact, to find the new limits of integration we just draw the cross section for fixed θ again, taking out the vertical arrow, and putting in an arrow for fixed ϕ.



A ρ-arrow, for fixed ϕ, enters the region at $\rho = 0$ and leaves where $r = 3$, which is the same as where $\rho \sin\phi = 3$, so where $\rho = \frac{3}{\sin\phi}$. So if we put integration with respect to ρ innermost, the limits of integration will be from 0 to $\frac{3}{\sin\phi}$. Since the $z = r$ line is the same as $\phi = \frac{\pi}{4}$, the limits on the $\int d\phi$ will be from $\phi = \frac{\pi}{4}$ to $\phi = \frac{\pi}{2}$. Now that we've figured out limits of integration, let's back up and note that the cylindrical integrand (without the dV) is actually just 1, since $r\,dz\,dr\,d\theta$ is the cylindrical equivalent of dV. So the spherical integrand (without the dV should also be 1, and for the dV we will use $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. Our net result is the following spherical equivalent for the original integral:

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{3}{\sin\phi}} \rho^2\, d\rho\, d\phi\, d\theta.$$

(b) The original integral looks easier to evaluate. In fact the evaluation is quite straightforward. You should get 18π.

Problem 20, §16.5, p766. Write a triple (iterated) integral representing the volume of a slice of the cylindrical cake of height 2 and radius 5 between the planes $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$. Evaluate this integral.

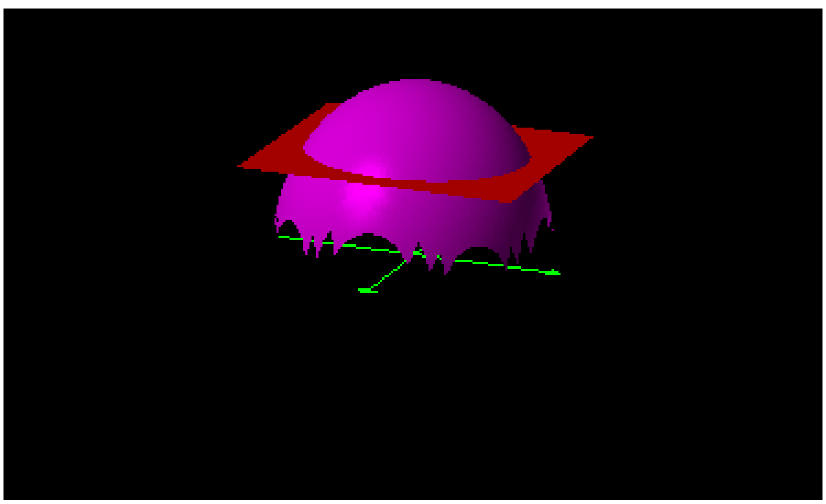
Solution. Apparently the z-axis is to be the central axis of the cake. We may as well put the bottom of the cake at $z = 0$. Use cylindrical coordinates. Cross sections for fixed θ will just be rectangles characterized by $0 \le z \le 2,\ 0 \le r \le 5$. For dV use the cylindrical equivalent $r\,dr\,dz\,d\theta$ (in some order). One possible integral for the volume of the cake is

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^2 \int_0^5 r\,dr\,dz\,d\theta = (\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta)(\int_0^2 dz)(\int_0^5 r\,dr) = (\frac{\pi}{6})(2)(\frac{25}{2}) = \frac{25\pi}{6}.$$

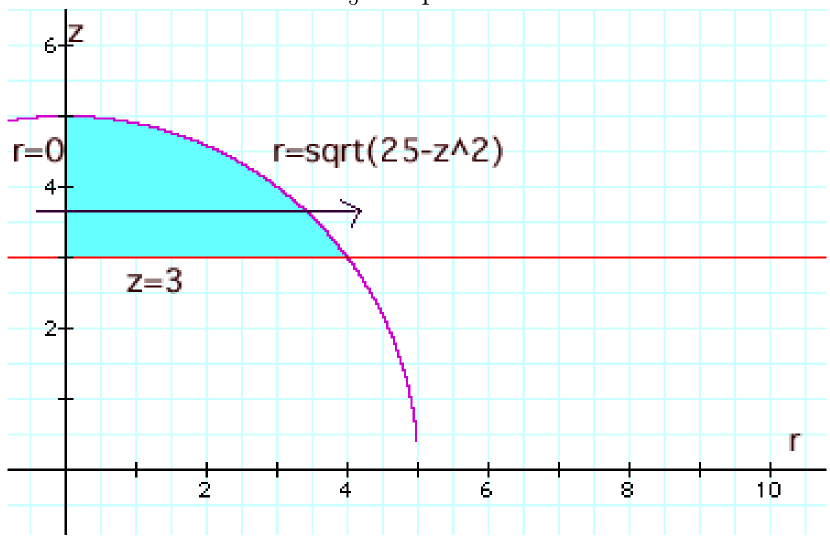
Other orders of integration are possible.

Problem 23, §16.5, p766. The figure in the text shows a part of a spherical ball of radius 5 cm, namely the portion lying between the bottom of the ball and a plane 2 cm above the bottom. (This is the same region as was used in HWK 19c, by the way.) Write an integral in cylindrical coordinates representing the volume of this region and evaluate it.

Solution. Refer to the solution for Problem 37, §16.3, HWK 19c for introducing a coordinate system. Here's the sketch we used before.



If we call our region W, we will be evaluating $\int_W 1\,dV$ using cylindrical coordinates. The constant function 1 remains 1, whatever coordinate system we use, but we must use $r\,dr\,dz\,d\theta$ (in some order or other) for dV. To find limits of integration we can either begin by shooting a z-arrow or begin by considering a cross section for fixed θ. I'll choose the latter, since the former is pretty similar to what we did before. The cross sections for fixed θ will look a little like a triangle, with one side along the z-axis (so where $r = 0$), one side along the half-line $z = 3$, and a third, curvilinear, side along the surface of the sphere, so where $r = \sqrt{25 - z^2}$.

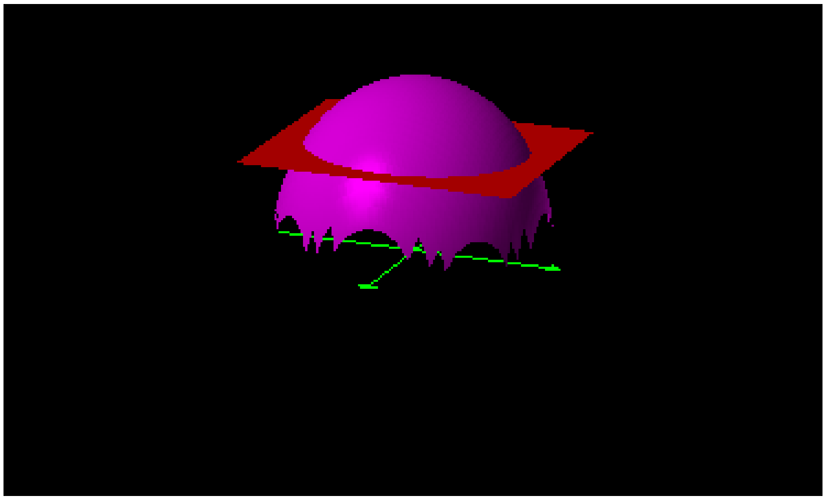


It will be easiest to shoot an r-arrow, for fixed z, as shown. The arrow enters the cross section where $r = 0$ and leaves where $r = \sqrt{25 - z^2}$. The z-values must vary from $z = 3$ to $z = 5$. Then θ must vary from 0 to 2π. This gives us, in cubic centimeters,

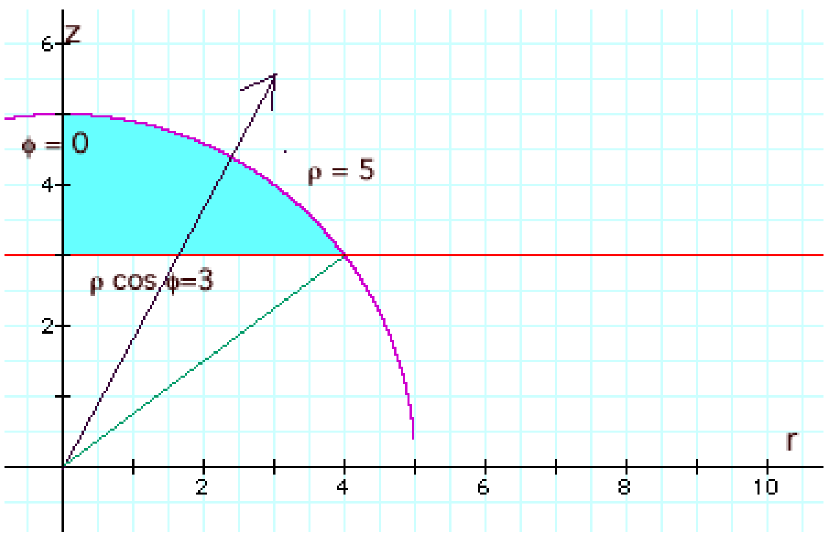
$$\begin{aligned}
\text{volume of } W &= \int_0^{2\pi} \int_3^5 \int_0^{\sqrt{25-z^2}} r\,dr\,dz\,d\theta \\
&= \int_0^{2\pi} \int_3^5 \left(\frac{25 - z^2}{2}\right) dz\,d\theta \\
&= \int_0^{2\pi} \left[\frac{25z}{2} - \frac{z^3}{6}\right]_3^5 \\
&= \int_0^{2\pi} \left(\frac{125}{2} - \frac{125}{6} - \frac{75}{2} + \frac{27}{6}\right) d\theta \\
&= \int_0^{2\pi} \frac{26}{3}\,d\theta \\
&= 14\pi \\
&\approx 54.45.
\end{aligned}$$

Problem 24, §16.5, p766. The figure in the text shows a part of a spherical ball of radius 5 cm, namely the portion lying between the bottom of the ball and a plane 2 cm above the bottom. (This is the same region as was used in Problem 23.) Write an integral in spherical coordinates representing the volume of this region and evaluate it.

Solution. Once again here's a computer-generated sketch of the region.



Just as we did for setting up the cylindrical-coordinate integral in Problem 23, we will examine the cross sections for fixed θ, only this time we will shoot a ρ-arrow for fixed ϕ, instead of an r-arrow or a z-arrow.



The formula for the spherical piece of the boundary becomes $\rho = 5$, that for the line $z = 3$ becomes $\rho \cos\phi = 3$, and that for the positive z-axis becomes $\phi = 0$. For fixed ϕ, a ρ-arrow will enter the region where $\rho = \frac{3}{\cos\phi}$ and leave where $\rho = 5$. What range of values must we use for ϕ?

We need to start at $\phi = 0$. The largest ϕ we need is determined by the angle formed between the positive z-axis and a ray from the origin to the point where the spherical curve hits the horizontal line $z = 3$. A little trigonometry gives us that this angle can be expressed as $\arccos \frac{3}{5}$. Finally we must let θ range from 0 to 2π, just as for cylindrical coordinates.

Remembering that we need to use $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ for dV, we find the spherical-coordinate integral for the volume of W to be:

$$\int_0^{2\pi} \int_0^{\arccos \frac{3}{5}} \int_{\frac{3}{\cos\phi}}^{5} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

I'm skipping the evaluation for now. It's reasonably straightforward. Value for the volume, in cubic centimeters is $\frac{52\pi}{3} \approx 54.45$.

Problem 25, §16.5, p766. For the region W shown in the text (the region is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the quadrant where $x \geq 0$, $y \geq 0$, while $z \leq 0$), write the limits of integration for $\int_W dV$ in the following coordinates:

(a) Cartesian (rectangular) (b) Cylindrical (c) Spherical

Solution. (a) A z-arrow will enter W where $z = -\sqrt{1 - x^2 - y^2}$ and leave where $z = 0$. Then (x, y) must vary over the quarter disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$. (Draw this quarter disk.) This gives

$$\int_W dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{0} dz \, dy \, dx.$$

Other correct answers are possible.

(b) This time look at a cross section for fixed θ. (Draw one. Label the boundaries using cylindrical coordinates.) It will be a quarter disk $z^2 + r^2 \leq 1$, $z \leq 0$. A z-arrow, say, would enter at $z = -\sqrt{1 - r^2}$. Then r would range from 0 to 1. The range of values for θ is 0 to $\frac{\pi}{2}$. So we get

$$\int_W dV = \int_0^{\frac{\pi}{2}} \int_0^1 \int_{-\sqrt{1-r^2}}^{0} r \, dz \, dr \, d\theta.$$

(c) Again, look at a cross section for fixed θ. (Draw one. Label the boundaries using spherical coordinates.) It will be a quarter disk with spherical boundary edge lying along $\rho = 1$, horizontal boundary edge $\phi = \frac{\pi}{2}$ and vertical boundary edge $\phi = \pi$. So a ρ-arrow enters at $\rho = 0$ and leaves at $\rho = 1$. Then ϕ must vary from $\phi = \frac{\pi}{2}$ to $\phi = \pi$. Finally, θ, as for cylindrical coordinates, must vary from 0 to $\frac{\pi}{2}$. The net result is

$$\int_W dV = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$