Mathematics 205
HWK 5a Solutions
Section 13.3 p627

Problem 5, §13.3, p627. Given

$$\vec{a} = 2\vec{j} + \vec{k} \quad \text{or} \quad \vec{a} = (0, 2, 1)$$

$$\vec{y} = 4\vec{i} - 7\vec{j} \quad \text{or} \quad \vec{y} = (4, -7, 0)$$

$$\vec{c} = \vec{i} + 6\vec{j} \quad \text{or} \quad \vec{c} = (1, 6, 0)$$

$$\vec{z} = \vec{i} - 3\vec{j} - \vec{k} \quad \text{or} \quad \vec{z} = (1, -3, -1)$$

find $(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z})$.

Solution. Since $\vec{a} \cdot \vec{y} = 0 \cdot 14 + 2 \cdot (-7) + 1 \cdot 0 = -14$ and $\vec{c} \cdot \vec{z} = 1 - 18 = -17$ we have

$$(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z}) = (-14)(-17) = 238.$$

Problem 6, §13.3, p627. Given $\vec{a}$ and $\vec{c}$ as in problem 5, find $((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a}$.

Solution. I would probably simplify first. Since $\vec{c} \cdot \vec{c}$ is a scalar, we can rewrite a little:

$$((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a} = (\vec{c} \cdot \vec{c})(\vec{a} \cdot \vec{a}).$$

Then find $\vec{c} \cdot \vec{c} = 37$ and $\vec{a} \cdot \vec{a} = 5$, so

$$((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a} = (\vec{c} \cdot \vec{c})(\vec{a} \cdot \vec{a}) = (37)(5) = 185.$$

Alternatively:

$$((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a} = (37\vec{a}) \cdot \vec{a} = 37(\vec{a} \cdot \vec{a}) = (37)(5) = 185.$$

Don't go to the effort of computing $37\vec{a}$ unless you enjoy doing arithmetic.

Problem 7, §13.3, p627. (a) Give a vector that is parallel to, but not equal to, the vector $\vec{v} = 4\vec{i} + 3\vec{j}$. (b) Give a vector that is perpendicular to $\vec{v}$.

Solution. (a) There are lots of correct answers. For instance, we could use $2\vec{v}$ (i.e. use $8\vec{i} + 12\vec{j}$). If you'd rather, use $59\vec{v}$ or $-\pi e\vec{v}$. Any scalar multiple $\lambda\vec{v}$ with $\lambda \neq 1$ will do (although $0\vec{v}$ would be an unusual answer).

(b) Since it's not clear whether $\vec{v}$ is intended to be a 2-space vector or a 3-space vector, I'll consider both interpretations.
2-space solution. Look for $\vec{w} = a\vec{i} + b\vec{j}$ satisfying $\vec{v} \cdot \vec{w} = 4a + 3b = 0$. There are lots of possibilities. For instance, $\vec{w} = -3\vec{i} + 4\vec{j}$ would work.
3-space solution. This is the same idea, only now we want $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$ with $4a + 3b + 0 = 0$.

It doesn't matter what c is, as long as $4a + 3b = 0$. For instance, $w = -3\vec{i} + 4\vec{j} + 5\vec{k}$ would work, and so would $\vec{w} = -3\vec{i} + 4\vec{j}$, just to name two possibilities.

Problem 9, §13.3, p627. Find a normal vector to the plane $z = 3x + 4y - 7$.

Solution. From our analysis in class, we know that the plane $ax + by + cz = d$ has $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ or $\vec{n} = (a, b, c)$ as one possible normal vector. Rewriting $z = 3x + 4y - 7$ as $3x + 4y - z = 7$, we can read off one correct answer: $\vec{n} = 3\vec{i} + 4\vec{j} - \vec{k}$ or $\vec{n} = (3, 4, -1)$. Other correct answers are possible.

Problem 12, §13.3, p627. Compute the angle between the vectors $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

Solution. Let $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, let $\vec{w} = \vec{i} - \vec{j} - \vec{k}$, and denote the angle between $\vec{v}$ and $\vec{w}$ by θ. Then we know $\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}|| \cos\theta$. In this case, that gives us $\vec{v} \cdot \vec{w} = -1 = \sqrt{3}\sqrt{3}\cos\theta$, so $\cos\theta = -\frac{1}{3}$. Therefore $\theta = \cos^{-1}(-\frac{1}{3}) \approx 1.91$ radians or $\approx 109.5^o$.

Problem 13, §13.3, p627. Which pairs of the vectors $\sqrt{3}\vec{i} + \vec{j}$, $3\vec{i} + \sqrt{3}\vec{j}$, $\vec{i} - \sqrt{3}\vec{j}$ are parallel and which are perpendicular?

Solution. Writing $\vec{u} = \sqrt{3}\vec{i} + \vec{j}$, $\vec{v} = 3\vec{i} + \sqrt{3}\vec{j}$, and $\vec{w} = \vec{i} - \sqrt{3}\vec{j}$, we have $\vec{u} \cdot \vec{v} > 0$, $\vec{u} \cdot \vec{w} = 0$, and $\vec{v} \cdot \vec{w} = 0$. This tells us that the pairs $\vec{u}, \vec{w}$ and $\vec{v}, \vec{w}$ are each perpendicular (hence not parallel), but $\vec{u}$ and $\vec{v}$ are not perpendicular to each other. Looking more carefully at $\vec{u}$ and $\vec{v}$, see that $\vec{v} = \sqrt{3}\vec{u}$, which shows that $\vec{u}$ and $\vec{v}$ are parallel.

Problem 14, §13.3, p627. For what values of t are $\vec{u} = t\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = t\vec{i} + t\vec{j} - 2\vec{k}$ perpendicular? Are there values of t for which they are parallel?

Solution. We have $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0 \iff t^2 - t - 2 = 0 \iff (t-2)(t+1) = 0 \iff t = 2, -1$.

For $\vec{u}$ and $\vec{v}$ to be parallel, we would need to have either $\vec{v} = \lambda\vec{u}$ for some scalar λ or $\vec{u} = \lambda\vec{v}$ for some scalar λ. The first situation would give $(t, t, -2) = (\lambda t, -\lambda, \lambda)$, which works out to be impossibile. Similarly, the second situation would require $(t, -1, 1) = (\lambda t, \lambda t, -2\lambda)$, which is also impossibile. So, no, it is not possible for $\vec{u}$ and $\vec{v}$ to be parallel, no matter how t is chosen.

Problem 17, §13.3, p627. Find an equation for the plane that is perpendicular to $5\overrightarrow{i}+\overrightarrow{j}-2\overrightarrow{k}$ and passes through the point $(0,1,-1)$.

Solution. We need only use the equation $\overrightarrow{n}\cdot(\overrightarrow{P_0P})=0$ taking $\overrightarrow{n}=5\overrightarrow{i}+\overrightarrow{j}-2\overrightarrow{k}$, $P=(x,y,z)$, and $P_0=(0,1,-1)$. This gives us the equation

$$5x+1(y-1)-2(z-1)=0$$

or

$$5x+y-2z=3.$$

Problem 19, §13.3, p627. Find an equation for the plane that passes through the point $(1,0,-1)$ and is is parallel to the plane $2x+4y-3z=1$.

Solution. Using $\overrightarrow{n}=2\overrightarrow{i}+4\overrightarrow{j}-3\overrightarrow{k}$ and $P_0=(1,0,-1)$, find the equation

$$2(x-1)+4y-3(z+1)=0$$

or

$$2x+4y-3z=5.$$

Problem 21, §13.3, p627. A certain plane has equation $z=5x-2y+7$.
(a) Find a value of λ making $\lambda\overrightarrow{i}+\overrightarrow{j}+(0.5)\overrightarrow{k}$ normal to the plane.
(b) Find a value of a so that the point $(a+1,a,a-1)$ lies on the plane.

Solution. (a) Rewrite the equation of the plane as $5x-2y-z=-7$. Read off the normal vector $\overrightarrow{n}=5\overrightarrow{i}-2\overrightarrow{j}-\overrightarrow{k}$. Now we need to choose λ so that $\overrightarrow{n}$ and $\lambda\overrightarrow{i}+\overrightarrow{j}+(0.5)\overrightarrow{k}$ are parallel. One way to find λ is to solve the proportionality condition $\dfrac{\lambda}{5}=\dfrac{1}{-2}=\dfrac{\frac{1}{2}}{-1}$. This gives $\lambda=-\dfrac{5}{2}=-2.5$.

(b) Merely substitute $(x,y,z)=(a+1,a,a-1)$ into the equation of the plane and solve for a:

$$5(a+1)-2a-(a-1)=-7$$

$$5a-2a-a=-7-5-1$$

$$2a=-13$$

$$a=-\frac{13}{2}=-6.5$$

Problem 25, §13.3, p627. Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular, to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

Solution. One solution. I'm going to write $\vec{p}$ (thinking "projection") for the component of $\vec{a}$ that is parallel to $\vec{d}$, and I'll write $\vec{q}$ for the component that is perpendicular to $\vec{d}$. We have the formulas (see class notes, though we might have used different letters there)

$$\vec{p} = \frac{\vec{a} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d}, \quad \vec{q} = \vec{a} - \vec{p}$$

A bit of arithmetic then gives us

$$\vec{p} = \frac{-8}{21} \vec{d}$$

and

$$\vec{q} = \frac{1}{21}(79\vec{i} + 10\vec{j} - 118\vec{k}).$$

With these choices, we have $\vec{a} = \vec{p} + \vec{q}$, $\vec{p}$ and $\vec{d}$ parallel, and $\vec{q} \perp \vec{d}$, as required.

Alternative solution. Let $\vec{u} = \dfrac{1}{||\vec{d}||} \vec{d}$. Write $\vec{p}$ and $\vec{q}$ as in the first solution. Then we have the formulas

$$\vec{p} = (\vec{a} \cdot \vec{u})\vec{u}, \quad \vec{q} = \vec{a} - \vec{p}.$$

Now do some arithmetic (pretty similar to the earlier arithmetic but not literally the same) and get the same results as before.

Problem 30, §13.3, p627. A course has four exams, weighted 10%, 15%, 25%, 50%, respectively. The class averages on these exams are 75%, 91%, 84%, and 87%, respectively. What do the vectors $\vec{a} = (0.75, 0.91, 0.84, 0.87)$ and $\vec{w} = (0.1, 0.15, 0.25, 0.50)$ represent in terms of the course? Calculate the dot product $\vec{w} \cdot \vec{a}$. What does it represent in terms of the course?

Solution. The vector $\vec{a}$ is the vector of class averages on the 4 successive tests. Of course they're written as decimals rather than percentages. The vector $\vec{w}$ is the vector of weights for the tests, again written as decimals. So $\vec{w} \cdot \vec{a}$ is a weighted average: it is the weighted class average for the 4 tests given in the course, and its value is $\vec{w} \cdot \vec{a} = .8565$. The class average for the semester is 85.65%.

Problem 31, §13.3, p627. A consumption vector of three goods is defined by $\vec{x} = (x_1, x_2, x_3)$, where x_1, x_2, and x_3 are the quantities consumed of the three goods. A budget constraint is represented by an equation $\vec{p} \cdot \vec{x} = k$, where $\vec{p}$ is the price vector for the three goods and k is a constant. Show that the different between two consumption vectors corresponding to points satisfying the same budget constraint is perpendicular to the price vector $\vec{p}$.

Solution. Let $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$ be two consumption vectors that correspond to points satisfying the same price constraint (say the constraint with price vector $\overrightarrow{p}$ and total budgeted amount k). By definition $\overrightarrow{p} \cdot \overrightarrow{x} = k$ and $\overrightarrow{p} \cdot \overrightarrow{y} = k$. We must show that $(\overrightarrow{x} - \overrightarrow{y}) \perp \overrightarrow{p}$, for which it suffices to show that $(\overrightarrow{x} - \overrightarrow{y}) \cdot \overrightarrow{p} = 0$, which we can verify as follows:

$$(\overrightarrow{x} - \overrightarrow{y}) \cdot \overrightarrow{p} = \overrightarrow{p} \cdot (\overrightarrow{x} - \overrightarrow{y}) = \overrightarrow{p} \cdot \overrightarrow{x} - \overrightarrow{p} \cdot \overrightarrow{y} = k - k = 0.$$