Mathematics 205
HWK 5b Solutions
Section 13.4 p634

Problem 5, §13.4, p634. Given $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$, use the algebraic definition of cross product to find $\vec{v} \times \vec{w}$.

Solution.

$$\begin{aligned}
\vec{v} \times \vec{w} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \\
&= \det \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} \vec{i} - \det \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \vec{j} + \det \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} \vec{k} \\
&= \vec{i} - (-3)\vec{j} + 7\vec{k} \\
&= \vec{i} + 3\vec{j} + 7\vec{k}
\end{aligned}$$

Problem 6, §13.4, p634. If $\vec{v} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$. What is the relation between the two answers?

Solution. Use the algebraic definition to find the specified cross products, expecting that the results will be negatives of each other. In fact,

$$\begin{aligned}
\vec{v} \times \vec{w} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & 2 & -1 \end{pmatrix} \\
&= \det \begin{pmatrix} -2 & 4 \\ 2 & -1 \end{pmatrix} \vec{i} - \det \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \vec{j} + \det \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix} \vec{k} \\
&= -6\vec{i} + 7\vec{j} + 8\vec{k}
\end{aligned}$$

while

$$\begin{aligned}
\vec{w} \times \vec{v} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & -2 & 4 \end{pmatrix} \\
&= \det \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix} \vec{i} - \det \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \vec{j} + \det \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} \vec{k} \\
&= 6\vec{i} - 7\vec{j} - 8\vec{k}
\end{aligned}$$

and $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$, as anticipated.

Problem 7, §13.4, p634. For $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$, find $\vec{a} \times \vec{b}$ and check that it is perpendicular to both $\vec{a}$ and $\vec{b}$.

Solution. The indicated cross product is

$$\begin{aligned}
\vec{a} \times \vec{b} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ 1 & -4 & 2 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & -1 \\ -4 & 2 \end{pmatrix} \vec{i} - \det \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \vec{j} + \det \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix} \vec{k} \\
&= -2\vec{i} - 7\vec{j} - 13\vec{k}
\end{aligned}$$

To check the perpendicularity, compute the relevant dot products:

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = (3\vec{i} + \vec{j} - \vec{k}) \cdot (-2\vec{i} - 7\vec{j} - 13\vec{k}) = -6 - 7 + 13 = 0$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = (\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (-2\vec{i} - 7\vec{j} - 13\vec{k}) = -2 + 28 - 26 = 0$$

Therefore $\vec{a} \perp (\vec{a} \times \vec{b})$ and $\vec{b} \perp (\vec{a} \times \vec{b})$, as expected.

Problem 9, §13.4, p634. Use the properties of cross product listed on p631 (anticommutativity, homogeneity, and left-distributivity of cross product over addition) to find $(\vec{i} + \vec{j}) \times (\vec{i} \times \vec{j})$.

Solution.

$$\begin{aligned}
(\vec{i} + \vec{j}) \times (\vec{i} \times \vec{j}) &= -[(\vec{i} \times \vec{j}) \times (\vec{i} + \vec{j})] = -[\vec{k} \times (\vec{i} + \vec{j})] \\
&= -[(\vec{k} \times \vec{i}) + (\vec{k} \times \vec{j})] = -[\vec{j} - (\vec{j} \times \vec{k})] = -\vec{j} + \vec{i} = \vec{i} - \vec{j}
\end{aligned}$$

Problem 13, §13.4, p634. If $\vec{v}$ and $\vec{w}$ are both parallel to the xy-plane, what can you conclude about $\vec{v} \times \vec{w}$? Explain.

Solution. If $\vec{v}$ and $\vec{w}$ are both parallel to the xy-plane, then $\vec{v} \times \vec{w}$ must be perpendicular to the xy-plane, since it must be perpendicular to each of $\vec{v}, \vec{w}$, hence to anything parallel to $\vec{v}, \vec{w}$. Since $\vec{v} \times \vec{w}$ must be perpendicular to the xy-plane, it must be parallel to the z-axis.

Incidentally, $\vec{v} \times \vec{w}$ could be the zero vector, which will happen if either $\vec{v}$ or $\vec{w}$ is the zero vector, or if $\vec{v}$ and $\vec{w}$ are parallel to each other.

Problem 15, §13.4, p634. Find an equation for the plane through the points $(3,4,2), (-2,1,0)$, and $(0,2,1)$.

Solution. Denote the given points by P, Q, and R, respectively, and let $\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. [In other words, let $\overrightarrow{n}$ denote the cross product of the displacement vector from $(3,4,2)$ to $(-2,1,0)$ with the displacement vector from $(3,4,2)$ to $(0,2,1)$.] Then $\overrightarrow{n}$ is one vector that is perpendicular to the desired plane. Computing $\overrightarrow{n}$, we find

$$\overrightarrow{n} = (-5\overrightarrow{i} - 3\overrightarrow{j} - 2\overrightarrow{k}) \times (-3\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k}) = (5\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}) \times (3\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}) = -\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$$

Choosing R to play the role of P_0, we get an equation for the plane:

$$(-\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}) \cdot (x\overrightarrow{i} + (y-2)\overrightarrow{j} + (z-1)\overrightarrow{k}) = 0$$

or

$$-x + (y-2) + (z-1) = 0$$

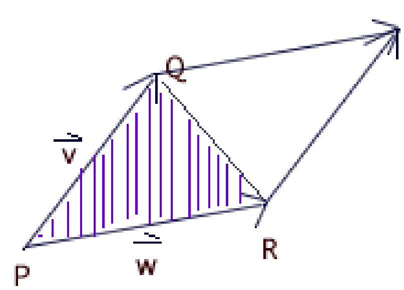
or simply

$$-x + y + z = 3.$$

In finding this equation, I made several choices (which displacement vectors to find cross product for, which point to use as a known point on the plane). Different choices will change the arithmetic along the way but will give the same equation in the end.

Problem 17, §13.4, p634. Let $P = (0,1,0), Q = (-1,1,2)$, and $R = (2,1,-1)$. Find
(a) The area of the triangle PQR.
(b) An equation for the plane that contains P, Q, and R.

Solution. (a)Let $\overrightarrow{v} = \overrightarrow{PQ}$ and $\overrightarrow{w} = \overrightarrow{PR}$. [In other words, let $\overrightarrow{v}$ be the displacement vector from the point $(0,1,0)$ to the point $(-1,1,2)$, and let $\overrightarrow{w}$ be the displacement vector from the point $(0,1,0)$ to the point $(2,1,-1)$]. Then the area we wish to find is the same as half the area of the parallelogram determined by $\overrightarrow{v}$ and $\overrightarrow{w}$. Here's a schematic diagram.



Using the geometric characterization of cross products, and writing A for the area of triangle PQR, we have

$$A = \frac{1}{2}||\overrightarrow{v} \times \overrightarrow{w}|| = \frac{1}{2}||(-\overrightarrow{i} + 2\overrightarrow{k}) \times (2\overrightarrow{i} - \overrightarrow{k})|| = \frac{1}{2}||3\overrightarrow{j}|| = \frac{3}{2} = 1.5.$$

(b) With $\overrightarrow{v}$ and $\overrightarrow{w}$ as in part (a), we know that $\overrightarrow{v} \times \overrightarrow{w}$ is perpendicular to the plane in question, and we already found that $\overrightarrow{v} \times \overrightarrow{w} = 3\overrightarrow{j}$.

Now, thinking geometrically, we can reason as follows. With $3\overrightarrow{j}$ as a normal vector, the plane we want has to be parallel to the xz-plane, hence has to be of the form $y = d$ for some constant d. Examining the points P, Q, R, we conclude that the specified plane has equation $y = 1$.

Alternatively, we can use $3\overrightarrow{j}$ as a normal vector and use, say, P as a known point on the plane. The usual formula shows the equation for the plane in question to be $-0(x-0)-3(y-1)+0(z-0) = 0$ or $3y = 3$, or simply $y = 1$.