Math 205
Quiz 2 Solutions

1. (20 points) Let $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, and let $\boldsymbol{d} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

(a) Find the magnitude of $\boldsymbol{d}$.

$$||\boldsymbol{d}|| = \sqrt{\boldsymbol{d} \cdot \boldsymbol{d}} = \sqrt{(2)(2) + (-1)(-1) + (1)(1)} = \sqrt{6}$$

(b) Find the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$. Denote the angle by θ. Then

$$\cos\theta = \frac{\boldsymbol{c} \cdot \boldsymbol{d}}{||\boldsymbol{c}||\ ||\boldsymbol{d}||} = \frac{2 - 1 + 2}{\sqrt{1+1+4}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

Therefore $\theta = \dfrac{\pi}{3}$ or $\theta = 60$ degrees.

(c) Give two different vectors that are perpendicular to $\boldsymbol{c}$ and two different vectors that are parallel to $\boldsymbol{c}$. There are lots of correct choices for both of these. For instance, $\boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}$ and $0\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ are both perpendicular to $\boldsymbol{c}$, since each one has zero dot product with $\boldsymbol{c}$. For two vectors parallel to $\boldsymbol{c}$ we could use, for instance, $2\boldsymbol{c} = 2\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ and $-\boldsymbol{c} = -\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$, since each one is a scalar multiple of $\boldsymbol{c}$.

(d) Find the component of $\boldsymbol{c}$ that is parallel to $\boldsymbol{d}$. The required vector is

$$\frac{\boldsymbol{c} \cdot \boldsymbol{d}}{\boldsymbol{d} \cdot \boldsymbol{d}}\boldsymbol{d} = \frac{3}{6}\boldsymbol{d} = \boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}.$$

(e) Find the component of $\boldsymbol{c}$ that is perpendicular to $\boldsymbol{d}$. The required vector is

$$\boldsymbol{c} - (\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} - (\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}) = \frac{3}{2}\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}.$$

(f) Find an equation for the plane that passes through the point $(1, 1, 1)$ and has $\boldsymbol{c}$ as a normal vector.
The equation is $1(x-1) + 1(y-1) + 2(z-1) = 0$ or

$$x + y + 2z = 4.$$

2. (6 points) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{3x^2+2y^2}$ does not exist. (Hint: try various straight-line paths.)
   At all points of the x-axis, and all points of the y-axis, the function in question has the value zero. So approach to $(0,0)$ along either of these axes will yield the limit 0. On the other hand

$$\lim_{(x,y)\to(0,0),x=y} \frac{4xy}{3x^2+2y^2} = \lim_{x\to 0} \frac{4x^2}{3x^2+2x^2} = \lim_{x\to 0} \frac{4x^2}{5x^2} = \lim_{x\to 0} \frac{4}{5} = \frac{4}{5}$$

   These different limit values (0 and $\frac{4}{5}$) prove that the specified limit does not exist.

3. (4 points) Let $f(x,y) = \dfrac{x^4+x^2y^2+y^4}{x^2+y^2}$.

   (a) Briefly tell why $0 \le f(x,y) \le \dfrac{x^4+2x^2y^2+y^4}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

The first inequality is true because the expression on the right-hand side of that inequality is a quotient with a positive numerator and a positive denominator. The second inequality is true because both sides of the inequality are nonnegative, both sides have the same denominator, and the right-hand side has a higher numerator.

   (b) Verify that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+2x^2y^2+y^4}{x^2+y^2} = 0$. (Hint: algebra.)

$$\lim_{(x,y)\to(0,0)} \frac{x^4+2x^2y^2+y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$$

   (c) What is the value for $\lim\limits_{(x,y)\to(0,0)} f(x,y)$? Zero.
   Why?

Except at $(0,0)$, we have

$$0 \le f(x,y) \le g(x,y) = x^2+y^2$$

with

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

Since the given function $f(x,y)$ is squeezed between two functions that have the same limit, namely zero, as $(x,y) \longrightarrow (0,0)$, the function $f(x,y)$ must also have zero limit as $(x,y) \longrightarrow (0,0)$.