Mathematics 206
Vector Subspaces

Here V is to be a vector space (with associated operations) and W is to be a subset of V.

What is a subspace?

Definition. We say that W is a **vector subspace** (or simply subspace, sometimes also called linear subspace) of V iff W, viewed with the operations it inherits from V, is itself a vector space.

Definition. We say that:
(a) W is **closed under addition** provided that $\boldsymbol{u}, \boldsymbol{v} \in W \Longrightarrow \boldsymbol{u} + \boldsymbol{v} \in W$
(b) W is **closed under scalar multiplication** provided that $\boldsymbol{u} \in W \Longrightarrow (\forall k \in \boldsymbol{R}) k\boldsymbol{u} \in W$.

In other words, W being closed under addition means that the sum of any two vectors belonging to W must also belong to W. Similarly, W being closed under scalar multiplication means that all scalar multiples of a vector belonging to W must also belong to W. For W to be a subspace of V, it must certainly be closed under both addition and scalar multiplication.

How can we decide whether W is a subspace?

As we discussed in class, deciding whether W is a subspace of V or not doesn't actually require us to check all 10 vector space axioms. It certainly suffices to check just VS1, VS6, VS4, and VS5, since the other axioms will hold automatically. In other words, it suffices to check that W is closed under addition and under scalar multiplication, that W has a zero vector, and that each vector belonging to W has a negative also belonging to W.

Even better, it suffices to just check that W is closed under addition and scalar multiplication and has the zero vector from V. For then axioms VS1, VS6, and VS4 hold. Moreover, if $\boldsymbol{u} \in W$, then $-\boldsymbol{u} = (-1)\boldsymbol{u} \in W$, and so VS5 holds as well.

Sometimes we can get by (very nearly, at least) with just checking that W is closed under addition and scalar multiplication. But then we need to know that W is at least nonempty (in other words that there is at least one vector belonging to W). If we do know W is nonempty and closed under addition and scalar multiplication, then the zero vector (for V) will also have to belong to W. Why? Since W is nonempty we may consider a fixed but unspecified vector $\boldsymbol{u} \in W$. For this particular vector, we know $0\boldsymbol{u} = \boldsymbol{0}$. Since W is closed under scalar multiplication, we conclude that $0\boldsymbol{u} \in W$ and therefore $\boldsymbol{0} \in W$.

Bottom line. If it's obvious that W is not empty, then we need only check for closure under addition and closure under scalar multiplication. If it's not obvious that W is nonempty, then we can get by with checking that the zero vector (for V) belongs to W and that W is closed under addition and closed under scalar multiplication.

Possible proof outlines for proving W is a subspace.

Outline 1, with detail.

(1) Check/observe that W is nonempty.

(2) Show that W is closed under addition.

(a) Let $\boldsymbol{u}, \boldsymbol{v} \in W$.

Ask yourself "what does this mean"?

Thus ask: what defining property/properties govern membership in W?

Thus assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy the membership property.

(b) Do some reasoning.

This is where the particular V and W enter in.

This is one of the only two parts of the proof outline that need new thought.

(c) Conclude that $\boldsymbol{u} + \boldsymbol{v} \in W$.

So show $\boldsymbol{u} + \boldsymbol{v}$ also satisfies the condition for membership in W.

(3) Show that W is closed under scalar multiplication.

(a) Let $\boldsymbol{u}, \in W$ and let k be a scalar (real number).

So assume $\boldsymbol{u}$ satisfies the membership property for W.

(b) Do some reasoning.

This is the other part of the proof outline that needs new thought.

(c) Conclude that $k\boldsymbol{u} \in W$.

So show $k\boldsymbol{u}$ satisfies the condition for membership in W.

Outline 1, without the detail.

(1) Check/observe that W is not empty.

(2) Show that W is closed under addition.

(3) Show that W is closed under scalar multiplication.

Outline 2.

(1) Check/show that the zero vector for V belongs to W.

(2) Show that W is closed under addition.

(3) Show that W is closed under scalar multiplication.

Possible proof outlines for showing W is not a subspace of V.

Outline 1. Show that the zero vector for V does not belong to W.

Outline 2. Exhibit two vectors belonging to W whose sum does not belong to W, thereby showing that W is not closed under addition.

Outline 3. Exhibit a vector $\boldsymbol{u} \in W$ and a scalar k for which $k\boldsymbol{u} \notin W$ (thereby showing that W is not closed under scalar multiplication).

Outline 4 (rarely used). Exhibit a vector $\boldsymbol{u} \in W$ for which $-\boldsymbol{u} \notin W$ (thereby showing that W fails axiom VS 5).

Note: For both Outline 1 and Outline 4, just above, there are a couple of subtle points that I'm not mentioning but that, strictly speaking, should perhaps be mentioned. I'm not mentioning them because I think they'll only add to any potential confusion and they can be settled so they don't really present potential problems. In other words, I'm acknowledging that there might be a little more one should say here without actually saying it. In some future course, you may need to deal with the more subtle points, but not here, and I don't want us to get bogged down in them.