Problem 5c, §3.1, p79. Write the number \( \sin(2i) \) in the form \( a + bi \).

Solution.

\[
\sin(2i) = \frac{1}{2i}[e^{i(2i)} - e^{-i(2i)}] = -\frac{i}{2}(e^{-2} - e^{2}) = \frac{(e^{2} - e^{-2})}{2}i = i \sinh 2
\]

Problem 5e, §3.1, p79. Write the number \( \sinh(1 + \pi i) \) in the form \( a + bi \).

Solution.

\[
\sinh(1 + \pi i) = \frac{e^{1+i\pi} - e^{-(1+i\pi)}}{2} = \frac{ee^{\pi i} - e^{-1}e^{-\pi i}}{2} = \frac{e(-1) - e^{-1}(-1)}{2} = -\frac{e - e^{-1}}{2} = -\sinh 1
\]

Problem 6 [for (8)], §3.1, p79. Using the definitions of \( \sin z \) and \( \cos z \) in terms of the exponential function, establish the identity \( \sin^{2} z + \cos^{2} z = 1 \) for all complex \( z \).

Solution. We have

\[
\sin^{2} z = \left( \frac{e^{iz} - e^{-iz}}{2i} \right)^{2} = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} = \frac{-e^{2iz} + 2 - e^{-2iz}}{4}
\]

and

\[
\cos^{2} z = \left( \frac{e^{iz} + e^{-iz}}{2} \right)^{2} = \frac{e^{2iz} + 2 + e^{-2iz}}{4}
\]

which gives

\[
\sin^{2} z + \cos^{2} z = \frac{1}{4}(2 + 2) = 1
\]

as required.

Problem 8 (for \( \sinh z \)), §3.1, p79. Verify (using the definition of \( \sinh z \) and previously known differentiation rules) that \( \frac{d}{dz} \sinh z = \cosh z \).

Solution.

\[
\frac{d}{dz} \sinh z = \frac{d}{dz} \frac{e^{z} - e^{-z}}{2} = \frac{1}{2} (e^{z} + e^{-z}) = \cosh z
\]
Problem 9b, §3.1, p79. Find $\frac{dw}{dz}$, given that $w = \cos(2z) + i\sin\left(\frac{1}{z}\right)$.

Solution.

$$\frac{dw}{dz} = \frac{d}{dz}\left(\cos(2z) + i\sin\left(\frac{1}{z}\right)\right) = -2\sin(2z) + i\left(\cos\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)\right) = -2\sin(2z) + \frac{-i}{z^2}\cos\left(\frac{1}{z}\right)$$

Problem 9e, §3.1, p79. Find $\frac{dw}{dz}$, given that $w = [\sinh(z) + 1]^2$.

Solution.

$$\frac{dw}{dz} = \frac{d}{dz}(1 + \sinh(z))^2 = 2(1 + \sinh(z))\cosh(z) = 2\cosh(z)(1 + \sinh(z))$$

Problem 11, §3.1, p79. Explain why the function $\text{Re}\left(\frac{\cos z}{e^z}\right)$ is harmonic in the whole plane.

Solution. The functions $\cos z$ and $e^z$ are entire, and $e^z$ is never zero. The quotient of analytic functions is analytic wherever the denominator is not zero, so the quotient $\frac{\cos z}{e^z}$ is entire. Therefore its real part, which is the given function $\text{Re}\left(\frac{\cos z}{e^z}\right)$, is harmonic in the whole plane. Alternatively, one could write the given function in terms of $x$ and $y$ and then compute its Laplacian to verify that the Laplacian is identically zero, but that’s much more work and probably less reliable.

Problem 13a, §3.1, p79. Show (using, for instance, the addition formula for the sine function), that $\sin(x + iy) = \sin x \cosh y + i\cos x \sinh y$.

Solution. First, an easy computation shows that $\cos(iy) = \cosh y$ and $\sin(iy) = i\sinh y$. Then the addition formula gives

$$\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy) = \sin x \cosh y + i\cos x \sinh y$$
as required.