Problem 1a, §3.2, 86. Evaluate \( \log i \).

Solution.

\[
\log i = \log |i| + i \arg i = \ln 1 + i \frac{\pi}{2} + 2k\pi i = i\left(\frac{\pi}{2} + 2k\pi\right) \quad (k \in \mathbb{Z})
\]

Problem 1c, §3.2, 86. Evaluate \( \text{Log} (-i) \).

Solution.

\[
\text{Log} (-i) = \text{Log} |-i| + i \text{Arg} (-i) = \ln 1 + i \frac{-\pi}{2} = -\frac{\pi}{2}i
\]

Problem 3, §3.2, 86. Show that if \( z_1 = i \) and \( z_2 = i - 1 \), then

\[
\text{Log} z_1 z_2 \neq \text{Log} z_1 + \text{Log} z_2.
\]

Solution. From \( z_1 z_2 = i(i - 1) = -1 - i \), we have

\[
\text{Log} z_1 z_2 = \text{Log} (-1 - i) = \text{Log} |-1 - i| + i \text{Arg} (-1 - i) = \text{Log} \sqrt{2} + i \text{Arg} \left(e^{-\frac{3\pi i}{4}}\right) = \ln \sqrt{2} - i \frac{3\pi}{4}
\]

while

\[
\text{Log} z_1 = \text{Log} i = \ln 1 + i \arg i = 0 + i \frac{\pi}{2} = \frac{\pi}{2}i
\]

\[
\text{Log} z_2 = \text{Log} (i - 1) = \text{Log} |i - 1| + i \text{Arg} (i - 1) = \ln \sqrt{2} + i \frac{3\pi}{4}i
\]

so

\[
\text{Log} z_1 + \text{Log} z_2 = \text{Log} (i) + \text{Log} (i - 1) = \frac{\pi}{2}i + \ln \sqrt{2} + i \frac{3\pi}{4}i = \ln \sqrt{2} + \frac{5\pi}{4}i
\]

Thus, as claimed,

\[
\text{Log} z_1 z_2 \neq \text{Log} z_1 + \text{Log} z_2.
\]

Problem 5c, §3.2, 86. Solve the equation \( e^{2z} + e^z + 1 = 0 \).

Solution. The equation is a quadratic equation in the expression \( e^z \). Using the quadratic
formula, we find $e^z = \frac{-1 + (1 - 4)^{\frac{1}{2}}}{2} = -\frac{1 + (-3)^{\frac{1}{2}}}{2}$. The two values for $(-3)^{\frac{1}{2}}$ are $\pm i\sqrt{3}$, so the two values for $e^z$ are

$$e^z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Using the inverse function (i.e. taking logarithms), we find

$$z = \log \left( -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right) = \log \left| -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right| + i \arg \left( -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right) = i \left( \pm \frac{2\pi}{3} + 2k\pi \right) \quad (k \in \mathbb{Z}).$$

**Problem 15, §3.2, 86.** Find a one-to-one analytic mapping of the upper half-plane $\text{Im} \ z > 0$ onto the infinite horizontal strip

$$H = \{ u + iv \mid -\infty < u < \infty, 0 < v < 1 \}$$

**Solution.** Give the name $U$ to the upper half-plane $\{ z \mid \text{Im} \ z > 0 \}$. Note that $0 < \arg z < \pi$ for $z \in U$. The function $w = \log z = \ln |z| + i \arg z$ maps $U$ in one-to-one analytic fashion onto the horizontal strip

$$S = \{ u + iv \mid -\infty < u < \infty, 0 < v < \pi \}.$$

We can modify this function slightly to get the mapping we want. In particular we map $U$ one-to-one onto $H$ by using the map $w = \frac{1}{\pi} \log z$.

**Problem 16, §3.2, 86.** Sketch the level curves for the real and imaginary parts of $\log z$ and verify that these level curves are orthogonal.

**Solution.** With $w = u + iv = \log z$, we have $u = \log |z| = \ln |z|$ and $v = \arg z$. Thus the level curves for $u$ are the curves $\ln |z| = \text{constant}$, and those for $v$ are $\arg z = \text{constant}$. The former are circles centered at the origin, and the latter are rays emanating from the origin. By a familiar theorem from Euclidean geometry, the rays from the origin are orthogonal to the circles centered at the origin, which gives us what we wished to show. Draw your own sketch.