Problem 3, §2.1, p 44. Describe the range of each of the following functions.

(a) \( f(z) = z + 5 \) for \( \text{Re} \, z > 0 \).

Solution. Since \( \text{Re} \, z > 0 \iff \text{Re} \, (z + 5) > 5 \), the range consists of those points in the \( w \)-plane that satisfy \( \text{Re} \, w > 5 \). In other words, the range consists of all the points that lie in the half-plane \( \text{Re} \, w > 5 \).

(b) \( g(z) = z^2 \) for \( z \) in the first quadrant, \( \text{Re} \, z \geq 0, \text{Im} \, z \geq 0 \).

Solution. Geometrically, we can think of \( g \) as squaring the modulus and doubling the argument. Since the specified domain could be specified by requiring \( 0 \leq \text{Arg} \, z \leq \frac{\pi}{2} \) the numbers \( w = g(z) \) belonging to the range will satisfy \( 0 \leq \text{Arg} \, w \leq \pi \). In other words, the range is the upper half-plane \( \text{Im} \, w \geq 0 \) (including the boundary line).

(c) \( h(z) = \frac{1}{z} \) for \( 0 < |z| \leq 1 \). Write \( h(z) = \frac{\pi}{|z|^2} \) and note that \( |h(z)| = \frac{1}{|z|} \). The points in the domain of \( h \) are those satisfying \( 0 < |z| \leq 1 \), so the points in the range are those satisfying \( |w| = |h(z)| \geq 1 \). In other words, the range consists of all the points in the \( w \)-plane that lie on or outside the unit circle.

(d) \( p(z) = -2z^3 \) for \( z \) in the quarter-disk \( |z| < 1, 0 < \text{Arg} \, z < \frac{\pi}{2} \).

Solution. We can think of \( p(z) \) as follows: starting with \( z \), cube the modulus, triple the argument, stretch by a factor of 2, and then reflect through the origin (in order to take the opposite or negative). Beginning with the specified quarter-disk, cubing the modulus gives us a quarter of the disk \( |w| < 1 \), namely the portion of \( |w| < 1 \) that lies in the first quadrant (without boundary edges). Tripling the argument then gives us three quarters of a disk, the three quarters that lie in the first three quadrants of the unit disk in the \( w \)-plane. Stretching by a factor of 2 gives us three quarters of the disk \( |w| < 2 \), namely the portion of this disk that lies in the first three quadrants. Finally, when we reflect through the origin, we end up with the three quarters of the disk \( |w| < 2 \) that lie in the first, third, and fourth quadrants of the \( w \)-plane. As the answer in the text states, the range can therefore be described by the conditions \( |w| < 2, -\pi < \text{Arg} \, w < \frac{\pi}{2} \).

Problem 7c, §2.1, p44. (for 208). A function of the form \( F(z) = z + c \), where \( c \) is a complex constant, generates a translation mapping. Sketch the image of the semidisk \( |z| \leq 2, \text{Im} \, z \geq 0 \) under \( F \) when \( c = -1 - i \).

Solution. We have \( F(z) = z + (-1 - i) \), with \( z \) in the specified semidisk. The function \( F \)
translates each \( z \) one unit to the left and one unit down. Therefore the image of the specified semidisk is another semidisk. The center of the disk is at the point \(-1 - i\), the radius is 2, and the semidisk lies above the horizontal line \( \text{Im} w = -1 \).

**Problem 8c, §2.1, p44.** (for 208). A function of the form \( G(z) = e^{i\phi}z \), where \( \phi \) is a real constant, generates a rotation mapping. Sketch the image of the semidisk \( |z| \leq 2, \text{Im} z \geq 0 \) under \( G \) when \( \phi = \frac{3\pi}{4} \).

**Solution.** In this case, \( G \) increases the argument of the input number \( z \) by \( \frac{3\pi}{4} \), which causes a counterclockwise rotation through an angle of \( \frac{3\pi}{4} \). So the image of the semidisk \( |z| \leq 2, \text{Im} z \geq 0 \) under \( G \) will be another semidisk of radius 2, centered at the origin, but with the diameter edge lying along the line \( \text{Im} w = -\text{Re} w \) (inclined at an angle of \( \frac{3\pi}{4} \) from the horizontal) rather than along the real axis in the \( w \)-plane.

**Problem 9, §2.1, p44.** (for 208). A function of the form \( H(z) = \rho z \), where \( \rho \) is a positive real constant, generates a magnification mapping when \( \rho > 1 \) and a reduction mapping when \( \rho < 1 \). Sketch the image of the semidisk \( |z| \leq 2, \text{Im} z \geq 0 \) under \( H \) when (a) \( \rho = 3 \) (b) \( \rho = \frac{1}{2} \).

**Solution.** With \( \rho = 3 \), the mapping \( H \) will be a magnification by a factor of 3. The specified semidisk will be magnified to yield a semidisk of radius 6, lying above the real axis in the \( w \)-plane and having vertices at \( w = \pm 6 \). With \( \rho = \frac{1}{2} \), \( H \) will be a reduction, shrinking \( z \) to half its original size. The image of the specified semidisk will still be a semidisk that touches and lies above the real axis in the \( w \)-plane, but the radius will be 1, so the two vertices will be at \( w = \pm 1 \).

**Problem 10, §2.1, p44.** (for 208). let \( F(z) = z + i \), \( G(z) = e^{i\pi/4}z \), and \( H(z) = \frac{z}{2} \). Sketch the image of the semidisk \( |z| \leq 2, \text{Im} z \geq 0 \) under each of the following composite mappings.

(a) \( G(F(z)) \)
Beginning with an input number \( z \), \( F \) translates the given \( z \) up one unit, and then \( G \) rotates it counterclockwise about the origin through an angle of \( \frac{\pi}{4} \). But the function \( G(F(z)) \) is actually easier to analyze if we rewrite it as \( G(F(z)) = e^{i\pi/4}z + e^{i3\pi/4} \), which we can view as a counterclockwise rotation through an angle of \( \frac{3\pi}{4} \), followed by a translation. The image is still a semidisk of radius 2 but the center is at the point \( \text{cis} \left( \frac{3\pi}{4} \right) \) and the diameter edge is tilted at an angle of \( \frac{\pi}{4} \) counterclockwise from the horizontal.

(b) \( G(H(z)) \)
Since \( H \) reduces by a factor of 2 and then \( G \) rotates counterclockwise by an angle of \( \frac{\pi}{4} \), the image will be a semidisk of radius 1, tilted so that the diameter line lies along \( \text{Re} \ w = \text{Im} \ w \).

(c) \( H(F(z)) \)
We know \( F \) translates up one unit, then \( H \) reduces by a factor of 2. So the image is a semidisk of radius 1, with center at the point \( \frac{i}{2} \) and the diameter line horizontal.

(d) \( F(G(H(z))) \)
First \( H \) reduces by a factor of 2, then \( G \) rotates counterclockwise by an angle of \( \frac{\pi}{4} \), then \( F \) translates up one unit. So the image is a semidisk of radius 1, centered at \( w = i \), and tilted counterclockwise from the horizontal by an angle of \( \frac{\pi}{4} \).

**Problem 11, §2.1, p45.** (for 310). Let \( F(z) = z - 3, G(z) = -iz, \) and \( H(z) = 2z \). Sketch the image of the circle \(|z| = 1\) under each of the following composite mappings.

(a) \( G(F(z)) \)
Before we start, note that \( F \) translates to the left by 3 units, \( G \) rotates clockwise about the origin through an angle of \( \frac{\pi}{2} \), and \( H \) magnifies by a factor of 2. So for \( G(F(z)) \) the image of the unit circle will be another circle of radius 1, but centered at \( 3i \).

(b) \( G(H(z)) \)
This time the image will be a circle of radius 2, still centered at the origin.

(c) \( H(F(z)) \)
This time the image will be a circle of radius 2, with center at \( w = -6 \).

(d) \( F(G(H(z))) \)
The image is a circle of radius 2, centered at \( w = -3 \).