**Problem 1c, §2.4, p62.** Use the Cauchy-Riemann equations to show that the function $w = 2y - ix$ is nowhere differentiable.

**Solution.** Writing $w = u + iv$, we have $u(x, y) = 2y$ and $v(x, y) = -x$. This gives $u_y = 2$ everywhere and $v_x = -1$ everywhere. At each point of $C$ we therefore have $u_y \neq -v_x$. The second of the two Cauchy-Riemann equations is not satisfied anywhere. The given function is not differentiable anywhere.

**Problem 2, §2.4, p62.** Show that $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but is nowhere analytic.

**Solution.** It suffices to prove that the given function is differentiable at each point of the coordinate axes but not anywhere else.

Writing $w = h(z) = u(x, y) + iv(x, y)$, we have $u(x, y) = x^3 + 3xy^2 - 3x$ and $v(x, y) = y^3 + 3x^2y - 3y$. This gives $u_y = 6xy$ and $v_x = 6xy$. In order to have $u_y = -v_x$, we must have $12xy = 0$ and therefore $x = 0$ or $y = 0$. Thus the only points where $h$ could possibly be differentiable are the points on the coordinate axes (where $x = 0$ or $y = 0$).

Next we show that $h$ actually is differentiable at the points on the coordinate axes. The easiest way is to use what the text calls Theorem 5 on p59. It’s easy to check that the other CRE, $u_x = v_y$, is satisfied everywhere. Moreover the partial derivatives of $u$ and $v$ are each continuous at every point of $C$. Therefore at each point of the coordinate axes, the hypotheses of Theorem 5 hold (partials exist in a neighborhood of the point, partials are continuous at the point, and partials satisfy CREs at the point). By Theorem 5, $h$ is differentiable at each point of the coordinate axes.