From
Steven Krantz's "How to teach mathematics"

CHAPTER I

Guiding Principles

1. Respect

You cannot be a good teacher if you do not respect yourself. If you are going to stand up in front of thirty people or three hundred people and try to teach them something, then you had better

- Believe that you are well qualified to do so.
- Want to do so.
- Be *prepared* to do so.
- Make sure that these characteristics are evident to your audience.

It is a privilege to stand before a group of people—whether they be young adults or your own peers—and to share your thoughts with them. You should acknowledge this privilege by (a) dressing appropriately for the occasion, (b) making an effort to communicate with your audience, (c) respecting the audience's point of view.

One completely obvious fact is this: you should have your material *absolutely mastered* before you enter the classroom. If you *do* possess this mastery, then you can expend the majority of your effort and attention on conveying the material to the audience. If, instead, you have a proof or an example that is not quite right, and if you stand in front of the group trying to fix it, then you will lose all but the die-hards quickly.

It is easy to rationalize that if the students were more able then they could roll with the ups and downs of your lecture. This is strictly illogical. How do you behave when you are listening to a colloquium or seminar and the lecturer goes off into orbit—either to fix an incorrect argument or into a private conversation with his buddy in the front row or, worse, into a private conversation with himself? All right then, now that you have admitted honestly how *you* behave, then can you really expect unseasoned freshmen to be tolerant when you do not seem to be able to do the examples that *they* are expected to do?

One of the best arguments for even elementary college mathematics courses to be taught by people with advanced degrees is this: since the material is all trivial and obvious to the august professor then the professor can maintain a broad sense of perspective, will not be thrown by questions, and can concentrate on the act of *teaching*.

If you respect yourself then, it follows logically, you will respect your audience: You should prepare your lecture. That way you will not be surprised by gaps in your thinking, you will not have to cast around for a necessary idea, you will not lose your train of thought in the lecture.

Throughout this booklet, I will repeatedly exhort you to prepare your lectures. I do not necessarily have in mind that you should spend an inordinate amount of *time* preparing. Consider by analogy the psychology of sport. Weight lifters, for example, are taught to meditate in a certain fashion before a big lift. Likewise, preparing is a way to collect your thoughts and put yourself in the proper frame of mind to give a lecture. It makes good sense. See also Section 1.4.

To me, preparation is the core of effective teaching. While this may sound like a tautology, and not worth developing, there is in fact more here than meets the eye. Just as being a bit organized relieves you of the stress and nuisance of spending hours looking for a postage stamp or a pair of scissors when you need them, so being the master of your subject gives you the ability to cope with the unexpected, to handle questions creatively, and to give proper stimulus to your students. An experienced and knowledgeable teacher who is comfortable with his/her craft is constantly adjusting the lecture, in real time, to suit the expressions on the students' faces, to suit their responses to queries and prods, to suit the rate and thoroughness with which they are absorbing the material. Just as a good driver is constantly making little adjustments in steering in response to road conditions, weather, and so forth, it is also the case that a good teacher (just as unconsciously) is engaging in a delicate give and take with the audience. Complete mastery is the unique tool that gives you the freedom to develop this skill.

You should treat questions with respect. I go into every class that I teach knowing full well that I am probably much smarter, and certainly much better informed, than most of the people in the room. But I do not need to use a room full of eighteen year olds as a vehicle for bolstering my ego. If a student asks a question, even a stupid one, then I treat it as an event. A wrong question can be turned into a good one with a simple turn of phrase from the instructor. If the question requires a lengthy answer then give a short one and encourage the student to see you after class. If you insult—even gently—the questioner then you not only offend that person but perhaps everyone else in the room. Once the students have turned hostile it is difficult to win them back, both on that day and on subsequent days.

If a student asks for permission to hand a paper in late, and you are tempted to say "I've set the deadline and the deadline is the deadline and how dare you ask me for an extension," then you should pause. You should ask yourself what you will have accomplished with this little speech. Does it really make any difference if this particular student hands the homework paper in tomorrow morning? If it does not, then is there any reason not to grant the extension? Is it possible that, given a little more time, this student will learn something extra? This consideration is just showing the student the sort of respect that you would have liked to have been shown when you were a student.

If a student wants to discuss why an exam was graded the way that it was, or why so many points were deducted on problem 5, or why he/she is not doing as well in the class as anticipated, then don't hide behind your rank. If you are not prepared to say a sentence or two about why you graded problem 5 the way that you did, it probably means that you graded the problem very sloppily and are afraid that you will be made to look foolish. It may mean that you are uncomfortable with confrontational scenes. But confrontational scenes may be avoided almost 100% of the time. If you are looking over problem 5 with the student, and if it turns out that the problem was basically right but you only gave 3 points out of 10, then you might say "I guess I read this problem too quickly. I get bleary-eyed late at night after reading sixty papers." The student is usually so grateful for the extra points that nothing more need be said.

To that (extremely) unusual student who says

> You are an incompetent boob. I am going to complain to the chairman of the department.

you could say "That is your privilege. Let me phone his/her secretary and make an appointment for you." Most of the time the student will back down. In the rare instances when the student does not retreat, any good chairman will give you ample opportunity to clarify the situation and smooth things over.

I don't mean to suggest in these pages that teaching is a confrontational activity. On the contrary, it can and should be a nurturing activity. But the potential for conflict is there, and I shall avail myself of several opportunities to suggest how you can either avoid or ameliorate conflict.

Another aspect of showing respect for your students is not springing nasty surprises on them. Do not say that you will grade a course one way and then grade it in a different fashion altogether. Do not act like an arrogant autocrat. (The truth is that, as a college teacher, you *are* an autocrat and a monarch and can do pretty much as you please. But there is no need to flaunt this before your students.) Do not create examinations that are full of dirty trick questions. If you do, you will be setting up a self-fulfilling situation in which everyone will do badly. Then you can brag to your pals that the students are stupid. How does this profit you? It is easier to make up a straightforward exam (not necessarily a watered-down exam) that tests students *on the material that you taught them.* If, as a result, the average on the test is 80% (a not very likely eventuality) then you can then tell the students how pleased you are that they have mastered the material so well.

If you give a miserable exam on which the average is 30% then you will have accomplished the following: for about 5% of the students (at the most) you will have set a standard to be risen to; but you will have alienated the other 95%. Students need not alienation but encouragement and (suitably tempered) challenge.

I have been known to complain to my colleagues that "students today do not know what it means to rise to a challenge". I was wrong to say this. It is true that today's students are not accustomed, as perhaps was a student like Bertrand Russell or G. H. Hardy, to have ever higher hurdles set before them at a prodigious rate. Many American students went through a grade school and a high school program in which they played a somewhat passive role. In a number of troubled high schools, students receive a grade of "A" or "B" provided only that they do not cause problems. This does not mean that students today are stupid. Rather, they may not have been challenged to rise to their full potential; they have never realized their capacity. In spite of television and other ostensibly insidious forces in our society, it is still a part of human nature to want to excel. It takes some practice to learn how to bring this out in students, but it can be done.

There are important philosophical and educational issues at play here. In America in the 1990s we attempt, as much as possible, to educate everyone. Whereas forty years ago this meant that "everyone" went to high school, now it means that "everyone" goes to college. It is vital in a free society that all citizens, regardless of financial resources, have an opportunity to pursue a college education. But society is set up in such a way now that a large percentage of young people go to college regardless of their interests or goals. For this we pay a price. We can rely less on the preparedness of our freshmen. We also can rely somewhat less on their attitudes and motivation.

What this means in practice is that, quite often, especially with freshmen—and especially at a public institution—we are not necessarily teaching a very select group. Many public institutions these days have an open admissions policy: anyone with a high school diploma has the right to attend the state university. From the taxpayers' point of view, such an admission policy makes perfectly good sense. If you are a professor at a state institution then you must make peace with the realities connected with such a policy. You must learn to adjust your expectations. You must learn a little patience, and learn to be flexible.

I am not about to recommend that college math teachers spend their evenings reading position papers on motivation such as are written at schools of education. I *am* recommending that the college mathematics teacher exercise some tolerance. Students *will* rise to a challenge, provided that the teacher starts with small challenges and works up to big ones. If students stumble at the first few challenges then they need encouragement, not derision. Exercising patience requires no more effort than exercising your vocal chords with an insulting remark.

The professor's attitude towards the class is apparent from his/her every word, every gesture, every action. If you are arrogant, if you despise your students, if you feel that you are above the task of teaching this course, then your students will get the message immediately. And what are you accomplishing by evincing these attitudes? Does it make you feel superior? More accomplished? More secure? More important? It should not. Proving a great theorem or writing a good book or article should make you feel secure and important and superior and accomplished. Doing a good job teaching the chain rule should make you feel as though you have done something worthwhile for someone else that day. There is something of value, of an intangible nature, about passing knowledge along to other people. Why not take some pleasure in it?

2. Attitude

I have long felt that those who cannot teach are those who do not care about teaching. If you actually care about transmitting knowledge then much of what I say in this booklet follows automatically. But some comments should be made.

Your students are a lot like you. When you enroll for a class, you have certain expectations. It is reasonable, therefore, that when you *teach* a class you should endeavor to live up to those same expectations. From this it follows that you should prepare, be organized, be fair, be receptive to questions, meet your office hour, and so forth.

On the other hand, your students are not like you. Especially in elementary courses, you cannot expect your students to be little mathematicians. Many of them are in the class *only* because it is a prerequisite for their major. Try to remember how you felt when you took anthropology or Latin or biology. Not everyone has a gift for mathematics. Unfortunately, some people have an attitude problem to boot (this attitude problem is sometimes termed "math anxiety"—see Section 1.13).

So you must learn to be sympathetic and receptive, and you must learn to be patient. Teaching is part of your craft, and part of your job. Perhaps if you are Gauss, or if you have just proved the Riemann hypothesis, then you can justifiably say that you are above these considerations. I'm betting that you are not either of these. If you call yourself a professor, and if you have the temerity to stand in front of an audience and profess, then you should show your audience some respect and consideration.

To stand in front of a class, with the charge of holding forth for an hour or more, is a heady experience—especially for the novice instructor. It is an ego trip. If you are prone to showing off anyway, this is an opportunity to let your predilections get out of hand. There is a temptation to tell too many jokes, to give a monologue, to use off-color language, to emphasize points with pratfalls or physical humor, to wear grotesque or offensive T-shirts or funny hats, to dress up like Isaac Newton, or to just be silly. A good rule of thumb is "Don't." There is a famous calculus teacher who used to wear a gorilla suit when he taught the chain rule. The idea was that the chain rule is so simple that even a monkey can do it. My view is that gimmicks such as this distract from the task at hand, which is to convey knowledge. How can a student concentrate on the mathematics if the instructor is dressed like a gorilla and acting silly to boot? If you want to wear a mask for the first couple of minutes of a lecture on Halloween, I guess that is all right. But do not introduce distractions into the classroom atmosphere.

For the length of a semester, you and your class are like a little family (if it is a large lecture, read "large, loosely knit, family"). The class develops its own *gestalt* and set of attitudes. Things will go smoothly if the attitude in your class is that you and the students are working together to conquer the material. If instead it is you and the book pitted against the students, then you've got an attitude problem. If, on the other hand, you take repeated pains to criticize the text, then you are setting up another attitude problem. No text is perfect; neither are the students and neither are you. But make it clear from the outset that you are on the students' side. You convey this attitude in thought, word, and deed: Prepare your lectures, respect student questions, give fair exams, meet your office hours.

Implicit in this discussion is a simple point: a successful class is not a confrontation between the professor and the students. The professor and the students should be allies, with the former playing the role of mentor, in mastering the material at hand. If that is not your role as teacher then what, pray tell, could it possibly be?

Your classes should be friendly, but you do not want to be friends with your students. This sounds a trifle cold, especially to a graduate student or new instructor, but it is an important device in maintaining control of the class. A slight distance helps preserve your authority. In particular, don't allow your students to call you "Bubbles" or any other affectionate name. It is probably not even a good idea to let your students address you by your first name.

Of course you should not date your students. It is safest not to date any student at your college or university. Given the way that students like to gossip about faculty, if you date one student you may as well be dating them all.

However, if you are inexorably smitten with one of your own students, then advise your love interest to transfer to another class or another section. Sexual harassment is an issue of great concern these days (Sections 3.7, 3.8). Professors are particularly vulnerable to charges of sexual harassment. Behave accordingly.

It is natural to want to show students your human side. There is probably no intrinsic harm in having a cup of coffee with some students. A relaxed atmosphere can help to open lines of communication. There might be some harm in having coffee with just one student—examine your conscience before doing this. If you meet a student in a bar off campus for a drink, then—let me speak frankly—you've got more on your mind than teaching.

Now let's return to teaching proper. A recurring theme in this booklet is that you should *prepare* your lectures. For a novice instructor, an hour or two or three of preparation may be necessary. For a seasoned trouper teaching calculus for the tenth time, as little as thirty minutes may be sufficient. The main thing is to be sure that you can do the calculations and that you have the definitions and theorems straight, and in the proper order. If you are the sort of person who freezes up in front of an audience, then be extra well-prepared. I know experienced professors who get so locked up in front of a large group that they cannot remember their own phone numbers. If that describes you, then have the necessary information on a sheet of paper.

If it is evident to your students that you are winging your lecture, then they are receiving a counterproductive message: if it is OK for the instructor to fake it then it is OK for the students to fake it. To those math teachers who say "I don't prepare because it is good for the students to see how a mathematician thinks" I say "nonsense." This is just laziness and/or self-serving rubbish. You must set a role model, both as an educated person and as to the way that mathematics is done.

The first few lectures that you give in a semester-long class set the tone for the entire semester. You may be at a slight disadvantage because you are coming off summer vacation. Perhaps you are not quite yet in the mood to be teaching, and your lack of enthusiasm shows. We've all fallen into the trap of saying, *sotto* *voce*, "I'll just wing the first few lectures and get things straightened out in a few weeks." This is a mistake. It is sending the wrong message to the class about your attitude towards discipline (both academic and personal), your attitude towards the students, and your attitude towards the subject matter. To repeat, you are a role model.

What your students write on their homework and on their exams will be a derivative version of what you show them. If you do not work out maximum-minimum problems systematically then they will not either. I always set up six steps to follow in doing a max-min problem and follow them scrupulously. This step-by-step approach is an elementary device, but it is an effective one for keeping interest up. When doing an example, after I do step two I can say "OK, so what do we do next" and this keeps the ball rolling.

Mathematicians fall unthinkingly into the use of jargon. Among ourselves, we frequently say 'trivially', 'clearly', 'by inspection', and so forth. Do not do this—especially in a calculus or pre-calculus class. First, it sounds pretentious. Second, it is dangerous to assume that anything is either trivial or clear unless you make it so. Third, to say that something is trivial is a subtle put-down: in the popular psycho babble this would be called "passive-aggressive behavior."

3. Personal Aspects

Like many activities in life, teaching is an intensely personal one. Some teachers have a lighthearted, informal, even jocular style. Others are more severe. Some give a rigid, structured lecture. Others conduct a Socratic interchange with the class. Some send students to the board to do problems (some, in the R. L. Moore style, do nothing but). Some instructors use overhead slides, computer simulations, symbolic manipulation software, and MATHEMATICA graphics. Others do it all themselves, with just a piece of chalk. Some professors integrate a (computer-based) laboratory component into their courses. [In fact I would like to see mathematics become more of a "laboratory discipline." But I defer a discussion of that topic to another time.] All of these methods are correct. It is essential for you to be comfortable with your class. Therefore you should conduct the class in whatever fashion feels most natural to you.

However you should be willing to try new things. If you have never told a joke before, try telling a joke. If it works, you may be pleasantly surprised and may tell another. [But be forewarned: eighteen year olds are insecure and are always worried that someone is making fun of them. Do not tell jokes that may be interpreted in that fashion. Do not tell jokes at anyone's expense. Do not tell sexist jokes. Do not use vulgar language.] Try introducing the product rule with a story about how much trouble Leibniz had getting it right. Illustrate the importance of the constant of integration by integrating $\int 1/x\,dx$ by parts (without the constant) and deriving the assertion that $0 = 1$:

$$\int \frac{1}{x}\,dx = \frac{1}{x} \cdot x - \int x \cdot \left(-\frac{1}{x^2}\right)\,dx = 1 + \int \frac{1}{x}\,dx$$

hence $0 = 1$. Some of these endeavors will fall flat. Others will breathe new life into an otherwise old and (for you) dull topic.

It is important to me that my classroom have the atmosphere of an interchange of ideas among intelligent people. I would be most uncomfortable to stand for an hour reciting a litany to a sea of blank faces. Thus I am continually trying new approaches, new angles, new ideas. It is a way to keep my lectures fresh, even in a course that I have taught ten times before.

I do not find it useful to send students to the blackboard to do problems. First, the time that it takes for the student to get to the front of the room, falter around, and sit down again, is too great for the benefit obtained. I do everything myself because I can teach a great deal even while I am doing the most mundane example. But others have been sending students to the board for years and swear by it. Do what works for you.

I have no use for overhead projectors. To me, part of pacing a lecture is letting it evolve on the blackboard. Part of the dynamic of my lecture is moving back and forth in front of the material. But others find that they can be more organized if they write out the material in advance on overhead slides. Still others write the material in real time on the overhead slide. Yet another group writes very little, but stands in one spot and delivers a strictly oral lecture.

Remember that you are delivering a product. Cadillac does this differently from Mercedes Benz. You must develop your own delivery. The over-riding consideration is that you be comfortable with your delivery so that you can make your class comfortable in turn. The style, organization, and content of your class is a reflection of you and your attitude towards the class. If you stand in front of your calculus class facing the blackboard, mumbling to yourself, and writing "Theorem–Proof–Theorem–Proof" then what message are you sending to the students? If instead you do a stand-up comedy routine and get around to the mathematics in the last ten minutes of class, then what message are you sending to the students?

Speaking strictly logically, it should not make any difference whether you wear a suit, or jeans and a work shirt, or wear a loin cloth and carry a spear when you teach. But it does. Dressing nicely sends a subtle signal to the class that you are the person in charge. Straightening your tie and combing your hair before going to class is like putting on your mortar board before going to graduation; you are pausing to say to yourself "now I am going off to do something important."

4. Prepare

Some people rationalize not teaching well by saying (either to themselves or to others) "My time is too valuable. I am not going to spend it preparing my calculus lecture. I am so smart that I can just walk into the classroom and wing it. And the students will benefit from watching a mathematician think on his feet." (As a student, I actually had professors who announced this to the class on a regular basis).

It is true that most of us can walk into the room most of the time and mostly wing it. But most of us will not do a very effective teaching job if we do so. Thirty minutes can be sufficient time for an experienced instructor to prepare a calculus lecture. A novice instructor, especially one teaching an unfamiliar subject for the first time, may need considerably more preparation time. Make sure that you have the definitions and theorems right. Read through the examples to make sure that there are no unpleasant surprises. It is a good idea to have a single page of notes containing the key points. To write out every word that you will say, write out a separate page of anticipated questions, have auxiliary pages of extra examples, have inspirational quotes drawn from the works of Thomas Carlyle, make up a new notational system, make up your own exotic examples, and so forth is primarily an exercise in self abuse. Over-preparation can actually stultify a lecture. But you've got to know your stuff.

I cannot over-emphasize the fact that preparation is of utmost importance if you are going to deliver an effective lecture or give a stimulating class. However it is also true that the more you prepare the more you lose your spontaneity. You must strike a balance between (i) knowing the material and (ii) being able to "talk things through" with your audience.

My own experience is that there is a "right amount" of preparation that is suitable for each type of course. I want to be confident that I'm not going to screw up in the middle of lecture; but I also want to be actually thinking the ideas through as I present them. I want to feel that my lecture has an edge. It *is* possible to over-prepare. To continue after you have prepared sufficiently is a bit like hitting yourself in the head with a hammer because it feels so good when you stop.

You must be sufficiently confident that you can field questions on the fly, can modify your lecture (again on the fly) to suit circumstances, can tolerate a diversion to address a point that has been raised. The ability to do this well is largely a product of experience. But you can *cultivate* this ability too. You cannot learn to play the piano by accident. And you will not learn to teach well by accident: you must be aware of what it is that you are trying to do and then consciously hone that skill.

If you do not prepare—I mean *really* do not prepare—and louse up two or three lectures in a row, then you will experience the following fallout: (i) students will take up your time after class and during your office hour (in order to complain and ask questions), (ii) students will complain to the undergraduate director and to the chairman, (iii) students will (if you are really bad) complain to the dean and write letters to the student newspaper, (iv) students will write bad teaching evaluations for your course.

Now student teaching evaluations are not gospel. They contain some remarks that are of value and some that are not. Getting bad teaching evaluations does not necessarily mean that you did a bad job. And I know that the dean will only slap me on the wrist if he gets a complaint about my teaching (however if there are ten complaints then I had better look out). Finally, I know that the chairman will give me the benefit of the doubt and allow me every opportunity to put any difficult situation in perspective. But if I spend thirty minutes preparing each of those three lectures then I will avoid all this grief and, in general, find the teaching experience pleasurable rather than painful. What could be simpler?

As well as preparing for a class, you would be wise to de-brief yourself after class. Ask yourself how it went. Were you sufficiently well prepared? Did you handle questions well? Did you present that difficult proof as clearly as you had hoped? Was there room for improvement? Be as tough on yourself as you would after any exercise that you genuinely care about—from playing the piano to a having a tennis match. It will result in real improvement in your teaching.

Read your teaching evaluations. Many are insipid. Others are puerile. But if ten of the students say that your writing is unclear, or that you talk too quickly, or that you are impatient with questions, then maybe there is a problem that you should address. Teaching is a yoga. Your mantra is "am I getting through to them?"

It is a good idea to try to anticipate questions that students will ask. But you cannot do this artificially, as a platonic exercise late at night over a cup of coffee. It comes with experience. Assuming that you have adopted the attitude that you actually care whether your students learn something, then after several years of teaching you will know by instinct what points are confusing and why. This instinct enables you to prepare a cogent lecture. It helps you to be receptive to student questions. It helps you to have a good attitude in the classroom.

An easy way to cut down on your preparation time for a lecture is to present examples straight out of the book. The weak students will appreciate this repetition. Most students will not, and you will probably be criticized for this policy. On the other hand, it is rather tricky to make up good examples of maximum-minimum problems or graphing problems or applications of Stokes's theorem. It can be time-consuming as well. A good rule of thumb is this: if you need more examples for your calculus class, pick up another calculus book and borrow some. Develop a file of examples that you can dip into each time you teach calculus. You will learn quickly that making up your own examples is hard work. Do you ever wonder why most calculus books are so disappointing? All right, *you* try to make up eight good examples to illustrate the divergence theorem.

5. Clarity

When you teach a mathematics class, clarity (or lack thereof) manifests itself in many forms. If you are the most brilliant, and even the most well-prepared, mathematics lecturer in the world, but you stand facing the blackboard and mumbling to yourself, then you are not being clear. If instead you shout at the top of your lungs so that all can hear, but if your handwriting is cryptic, then you are not being clear. If your voice is clear, your handwriting clear, but your blackboard technique non-existent, then you are not being clear. If your voice is beautiful, your handwriting artistic, your blackboard technique flawless, but you are completely disorganized, then you are not being clear. If you speak clearly, write clearly, have good blackboard technique, are well organized, but speak with a foreign accent, then don't worry. You are being clear.

Here is the point. Mathematics is hard. Do not make it harder by putting artificial barriers between you and your students. If you are shy and simply cannot face your audience, then perhaps you chose the wrong profession. More seriously, be very well prepared. *Make* yourself confident. Calculus is one of the most powerful analytic tools that has ever been created. It is a privilege to be able to pass it along to the next generation. Be proud of what you are doing. It is no less an event for you to teach the fundamental theorem of calculus to a group of freshmen in the 1990s than it was for Archimedes to teach his students how to calculate the area of a circle.

I have atrocious handwriting. When my departmental librarian got her first written message from me she thought it had been written in Chinese hieroglyphs. But when I lecture I slow down. I write deliberately. I *want* my audience to understand me and to respect me and I take steps to see that this actually happens.

Suppose that you are in the middle of a lecture and you are making a very important point. How can you drive it home? How can you get the students' attention? We all know that students drift into a malaise in which they are copying and not thinking (after all, we were once students and did the same). How do you wake them up? It's easy. Pause. State the point clearly and simply. Write it clearly and simply. Say "This is important." Repeat the point. One of Mozart's most effective tools in his compositions was to repeat a particularly beautiful passage. We can benefit from his example.

Ask whether there are any questions. Repeat the point again. Assure students that this point will be on the exam, and that it will come up over and over again in the course. Tell them that if they do not understand this point then it will hamper them later in the course. Knowing how to make certain that students know when you are making an important point is a big—and infrequently mentioned—aspect of the "clarity" issue.

If your teaching evaluations say that "the exam didn't cover the points stressed in class," it may mean that you don't know how to write a good exam (Section 2.9). But it may also mean that you don't know how to put your point across, or how to tell the students what is important—in other words, how to make yourself clear.

6. Speak up

If you are going to be a successful lecturer then you have to find a way to fill the room with yourself. If you stand in front of the class (be it a class of ten or a class of a hundred) and mumble to yourself then you will not successfully convey the information. Even the most dedicated students will have trouble paying attention. You will not have stimulated anyone to think critically.

You do not need to be a show-off or a ham or a joke teller to fill a room with your presence. You can be dignified and reserved and old-fashioned and still be a successful lecturer with today's students. But you must let the students know that you are there. You must establish eye contact. You must let them know that you are *talking to them.*

Before I start a lecture, especially to a large class, I engage some of the students in informal conversations. I get them to talk about themselves. I ask them how they are doing on the homework assignment. I comment about the weather. Then I make a smooth transition into the lecture. That way I already have half a dozen people on my side. The others soon follow.

Some new instructors—especially those who are naturally soft spoken or shy—may need some practice with voice modulation and projection. Get together a group of friends and give a practice lecture for them. Ask for their criticism. Make a tape recording of your practice lecture and listen to it critically.

If there is any doubt in your mind as to whether you are reaching your audience during a lecture, then *ask* about it. Say "Can you hear me? Am I talking loud enough? Are there any questions?" This is one of many simple devices for changing the pace of a lecture, giving note-takers a break, allowing students to wake up.

Think of a good movie that you've seen recently. Now remove the music; remove the changes in focal length; remove the changes of scenery; remove the voice modulation and changes of emotion; remove the skillful use of silence as a counterpoint to sound. What would remain? Could you stay awake during a showing of what is left of this movie? Now think about your class in these terms.

7. Lectures

In an empty room sits a violin.

One person walks in, picks it up, draws the bow across the strings, and a horrible screeching results. He leaves in bewilderment.

A second person walks in, attempts to play, and the notes are all off key.

A third player picks up to the instrument and produces heavenly sounds that bring tears to the eyes. He is Isaac Stern and the instrument is a Stradivarius.

Wouldn't it have been foolish to say, after hearing the first two players, that this instrument is outmoded, that it doesn't work? That it should be abandoned to the scrap heap? Yet this is what many are saying today about the method of teaching mathematics with lectures. Citing statistics that students are not learning calculus sufficiently well, or in sufficiently large numbers, government sponsored projects nationwide assert that the lecture doesn't work, that we need new teaching techniques.

Those who say that "the use of the lecture as an educational device is outmoded" rationalize their stance, at least in part, by noting that we are dealing with a generation raised on television and computers. They argue that today's students are too ready to fall into the passive mode when confronted with a television-like environment. It follows that we must teach them interactively, using computers and software to bring them to life.

Lectures have been used to good effect for more than 3000 years. I am hesitant to abandon them in favor of a technology (personal computers, videos) that has existed for just ten years. In spite of popular rumors to the contrary, a lecture does not need to be a bone dry desultory Philippic. It can have wit, erudition, and sparkle. It can arouse curiosity, inform, and amuse. It is an effective teaching device that has stood the test of time. The ability to give a good lecture is a valuable art, and one that you should cultivate.

However you really have to work at making your lectures reach your students. It is true that mathematics teaching in this country is not, overall, very effective. The reason, however, is not that the lecture method is "broken." Rather, we tend not to put a lot of effort into our teaching because the reward system is often not set up to encourage putting a lot of effort into it. You must learn to develop eye contact with your audience, to fill the room with your voice and your presence, and to present your ideas with enthusiasm and clarity. Other sections of this booklet deal in detail with these techniques.

Turn on your television and watch a self-help program, or a television evangelist, or a get-rich-quick real estate huckster. These people are not using overhead projectors, or computer simulations, or MATHEMATICA. In their own way they are lecturing, and *very effectively.* They can convince people to donate money, to change religions, or to join their cause. Of course your calculus lecture should not literally emulate the methods of any of these television personalities. But these people and their methods are living proof that the lecture is not dead, and that the traditional techniques of Aristotelian rhetoric are as effective as ever.

There are other useful teaching environments besides lectures. Although less common in mathematics than in some humanities courses, group discussions can be useful. If you want to get students interested in what the boundary of a set in a metric space ought to be, begin with a discussion in which students offer various suggestions. Before you define what a finite set is, ask the students to suggest a definition. See also Section 2.5 on the benefits of group activity.

As already mentioned, it can also be instructive to have students volunteer to do problems at the blackboard. Once in a great while, if a student offers an alternate proof of a proposition or another point of view, I hand him/her the chalk. Everyone is usually quite surprised, but the results are generally pleasing and it provides a nice change of pace.

Computer labs can also be a useful instructional device. The subject of sophomore level differential equations lends itself well to helping students explore the interface between what we can do by hand and what the machine can teach us. Let's be frank: we do not know how to solve most differential equations. Thus it is important for students to see how much analysis one can do with traditional methodologies and then to see how the machine can use phase plane analysis, numerical methods, and graphing to provide further concrete data.

It is important that we constantly seek new and better methods and technologies for teaching. This author, and this booklet, has a built in bias toward traditional methods, such as lectures. That is because he has watched them work and used them successfully for more than twenty years. I hope that other writings will explore some of the new teaching techniques.

8. Questions

In a programmed learning environment, whether the interface is with a PC or with MATHEMATICA notebooks or with a MAC, the student cannot ask questions. The give and take of questions and answers is a critical aspect of the human part of the teaching process. Teachers are *supposed* to answer questions.

There is more to this than meets the eye. When I say that a teacher answers questions I do not envision the student saying "What is the area of a circle?" and the teacher saying " πr^2." I instead envision the student struggling to articulate some confusion and the experienced teacher turning this angst into a cogent question and then answering it. To do this well requires experience and practice. I frequently find myself responding to a student by saying "let's set your question aside for a minute and consider the following." I then put the student at ease by quickly running through something that I know the student knows cold, and that serves as a setup for answering the original question. With the student on my side, I can answer the primary problem successfully. The point is that some questions are so ill-posed that they literally cannot be answered. It is the teacher's job to make the question an answerable one and then to answer it. See also Section 2.19 on asking and answering questions.

A similar, but alternative scenario is one in which the student asks a rather garbled question and I respond by saying "Let me play the question back for you in my own words and then try to answer it ... ." The point is that the responses "Your question makes no sense" or "I don't know what you mean" are both insulting and a cop-out. To be sure, it is the easy answer; but you will pay for it later. It takes some courage for the student to ask a question in class; by treating questions with respect, you are both acknowledging this fact and helping someone to learn.

Yet another encouraging response to a student question is to say "Thank you. That question leads naturally to our next topic ... " Of course you must be quick on your feet in order to be able to pull this off. It is worth the trouble: students respond well when they are treated as equals.

There are complex issues involved here. A teacher does not just lecture and answer questions. A good teacher helps students to discover the ideas. There are few things more stimulating and rewarding than a class in which the students are anticipating the ideas because of seeds that you have planted. The way that you construct your lecture and your course is one device for planting those seeds. The way that you answer questions is another.

When I discuss teaching with a colleague who has become thoroughly disenchanted with the process, I frequently hear complaints of the following sort: "Students these days are impossible. The questions that they pose are unanswerable. Suppose, for example, that I am doing a problem with three components. I end up writing certain fractions with the number 3 in the denominator. Some student will ask 'Do we always put a 3 in the denominator when doing a problem from this section?' How am I supposed to answer a question like that?"

Agreed, it is not obvious how to answer such a question, since the person asking it either (i) has not understood the discussion, (ii) has not been listening, or (iii) has no aptitude for the subject matter. It is tempting to vent your spleen against the student asking such a question. Do not do so. The student asking this question probably needs some real help with analytical thinking, and you cannot give the required private tutorial in the middle of a class hour. But you can provide guidance. Say something like "When a problem has three components it is logical that factors of 1/3 will come up. This can happen with certain problems in this section, or in any section. But it would be wrong to make generalizations and to say that this is what we do in all problems. If you would like to discuss this further, please see me after class." In a way, you are making the best of a bad situation; but at least you are doing something constructive, and providing an avenue for further help if the student needs it.

Let us consider some other illustrations of the principle of making a silk purse from a sow's ear:

The first example is a simple one.

Q: Why isn't the product rule $(f \cdot g)' = f' \cdot g'$?

The answer is *not* "Here is the correct statement of the product rule and here is the proof." Consider instead how much more receptive students will be to this:

A: Leibniz, one of the fathers of calculus, thought for many years that this is what the product rule should be. But he was unable to verify it. Of course Leibniz was hampered because he didn't have the language of functions. We do. If we set $f(x) = x^2$ and $g(x) = x$ then we can see rather quickly that $(f \cdot g)'$ and $f' \cdot g'$ are unequal. So the simple answer to your question is that the product rule that you suggest gives the wrong answer. Instead, the rule $(f \cdot g)' = f' \cdot g + g' \cdot f$ gives the *right* answer and can be verified mathematically.

The second example is more subtle.

Q: Why don't we divide vectors in three-space?

The *wrong* answer is to tell about Stiefel-Whitney classes and that the only Euclidean spaces with a division ring structure are $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4$, and $\mathbb{R}^8$. A better answer is as follows:

A: J. Willard Gibb invented vectors to model physical forces. There is no sensible physical interpretation of "division" of physical forces. The nearest thing would be the operations of projection and cross product, which we will learn about later.

Notice that in both illustrations an attempt is made to turn the question into more than what it is—to make the questioner feel that he/she has made a contribution to the discussion.

Q: Why isn't the concept of velocity in two and three dimensions a number, just like it is in one dimension?

If you are in a bad mood, you will be tempted to think that this person has been dreaming for the last hour and has understood absolutely nothing that you have been saying. Bear up. Resist the temptation to voice your frustrations. Instead try this:

A: Let me rephrase your question. Instead let's ask "Why don't we use vectors in one dimension to represent velocity just as we do in two and three dimensions?"

One of the most important features of vector language is that a vector has *direction* as well as magnitude. In one dimension there are only two directions: right and left. We can represent those two directions rather easily with either a plus or a minus sign. Thus positive velocity represents motion from left to right and negative velocity represents motion from right to left. The vector language is *implicit* in the way that we do calculus in one dimension, but we need not articulate it because positivity and negativity are adequate to express the directions of motion.

In dimensions two and higher there are infinitely many different directions and we require the explicit use of vectors to express velocity.

As the author of this booklet, I have the luxury of being able to sit back and think carefully about how to formulate these "ideal" answers to poor questions. When you are teaching you must be able to do this on your feet, either during your office hour or in front of a class. At first you will not be so articulate. This is an acquired skill. But it is one *worth acquiring.* It is a device for showing respect for your audience, and in turn winning its respect.

A final note about questions. Even though you are an authority in your field, there are things that you don't know. Occasionally these lacunae in your knowledge will be showcased by a question asked in class or during your office hour (it does not happen often, so don't get chills). The sure and important attribute of an intelligent, educated individual is an ability to say "I don't know the answer to that question. Let me think about it and tell you next time." On the (rare) occasions when you have to say this, be sure to follow through. If the item that you don't know is an integral part of the class—and this had better not be the case very often—get it down cold because the question is liable to come up again in a different guise later in the course. If it is not an integral part of the course, then you have no reason to feel badly. Just get it straight and report back.

The main point here is this: do not under any circumstances try to fake it. If you do, you will look bad, your interlocutor will be frustrated and annoyed, and you will have served no good purpose. If there is any circumstance in which honesty is the best policy, this is it. Professor of Economic History Jonathan R. T. Hughes was wise to observe that "There is no substitute for knowing what you are talking about."

9. Inductive vs. Deductive Method

It is of paramount importance, epistemologically speaking, for us as scholars to know that mathematics can be developed *deductively* from certain axioms. The axiomatic method of Euclid and Occam's Razor have been the blueprint for the foundations of our subject. Russell and Whitehead's *Principia Mathematica* is a milestone in human thought, although one that is perhaps best left unread. Hilbert and Bourbaki, among others, also helped to lay the foundations that assure us that what we do is logically consistent.

However mathematics, as well as most other subjects, is not learned deductively: it is learned *inductively.* We learn by beginning with simple examples and working from them to general principles. Even when you give a colloquium lecture to seasoned mathematicians, you should motivate your ideas with good examples. The principle applies even more assuredly to classes of freshmen and sophomores.

Take the fact that the mixed partial derivatives of a C^2 function in the plane commute. To state this theorem cold and prove it—before an audience of freshmen—is showing a complete lack of sensitivity to your listeners. Instead, you should work a couple of examples and then say "Notice that it does not seem to matter in which order we calculate the derivative. In fact there is a general principle at work here." Then you state the theorem.

Whether you actually give a proof is a matter of personal taste. With freshmen I would not. I'd tell them that when they take a course in real analysis they can worry about niceties like this. Other math instructors may feel differently about this point.

And by the way—you know and I know that C^2 is too strong a hypothesis for the commutation of derivatives. But really, isn't that good enough for freshmen? If a bright student raises this issue, offer to explain it after class. But do not fall into the trap of always stating the sharpest form of any given result. Great simplifications can result from the introduction of slightly stronger hypotheses, and you will reach a much broader cross-section of your audience by using this device.

Now suppose that you are teaching real analysis (from [RUD], for example). One of the neat results in such a course is that a conditionally convergent series can be rearranged to sum to any real limit. When I present this result, I first consider the series $\sum(-1)^j/j$ and run through the proof specifically for this example. The point is that, by specializing down to an example, I don't have to worry about proving first that the sum of the positive terms diverges and the sum of the negative terms diverges. That is self-evident in the example. Thus, on the first pass, I can concentrate on the main point of the proof and finesse the details.

Go from the simple to the complex—not the other way. It's an obvious point, but it works. Here is another example of that philosophy, implemented somewhat differently. Many calculus books, when they formulate Green's theorem, go to great pains to introduce the notions of "x- simple domain" and "y- simple domain" (i.e. domains with either simple horizontal or simple vertical cross-sections). This is because the authors are looking ahead to the proof, and want to state the theorem in precisely the form in which it will be proved. The entire approach is silly.

Why not state Green's theorem in complete generality? Then it is simple, sweet, and students can see what the principal idea is. When it is time for the proof, just say "to keep the proof simple, and to avoid technical details, we restrict attention to a special class of domains ... " This approach communicates exactly the points that you wish to convey, but cuts directly to the key ideas and will reach more of the students with less fuss.

Here is a useful device—almost never seen in texts or discussed in teaching guides—that was suggested to me by Paul Halmos:

Suppose that you are teaching the fundamental theorem of algebra. It's a simple theorem; you could just state it cold and let the students think about it. But the point is that these are *students*, *not* mathematicians. It is your job to give them some help. First present to them the polynomial equation $x - 7 = 0$. Point out that it is easy to find all the roots and to say what they are. Next treat $2x - 7 = 0$. Follow this by $x^2 + 2x - 7$ (complete the square—imitating the proof of the quadratic formula). Give an argument that $x^3 + x^2 + 2x - 7$ has a root by using the intermediate value property. Work a little harder to prove that $x^4 + x^3 + x^2 + 2x - 7$ has a root. Then surprise them with the assertion that there is no formula, using only elementary algebraic operations, for solving polynomial equations of degree 5 or greater. Finally, point out that the remarkable fundamental theorem of algebra, due to Gauss, guarantees in complete generality that any non-constant polynomial has a (complex) root.

Notice how much depth and texture this simple discussion lends to the fundamental theorem. You have really given the students something to think about. Stating the theorem cold and then moving ahead, while *prima facie* logical and adequate, does not constitute teaching—that is, it does not contribute to understanding. As with many of the devices presented in this booklet, this one becomes natural after some practice and experience. At the beginning it will require some effort. The easiest thing in the world for a mathematician to do is to state theorems and to prove them. It requires more effort to *teach*.

Beresford Parlett recently said

> Only wimps do the general case. Real teachers tackle examples.

I think that Halmos's ideas illustrate what Parlett is saying.

One could go on at length about the philosophy being promulgated here. But the point has been made. Saki once said that "A small inaccuracy can save hours of explanation." Mathematicians cannot afford to be inaccurate. But, for the students' sake, we can simplify.

10. Advanced Courses

The teaching problems that arise in an advanced course are rather different from those in a lower division course. You are dealing with a more mature audience and, in at least some advanced courses, many of your students will be math majors. The main message for a new teacher is: don't get carried away. Don't try to tell them about your Ph.D. thesis in the first week of class. Try to remember the troubles you had learning about uniform continuity and uniform convergence. Give lots of examples. Prepare your lectures well and *slow down*. Be receptive to questions and sympathetic to awkward struggles with sophisticated new ideas. Be willing to repeat yourself.

It is probably best in an advanced undergraduate course to cover less material but to cover it in depth—to endeavor to give the students a real feel for the subject—rather than to race through a lot of material. Again I shall repeat an implicit theme of this booklet: most undergraduate students do not have either the maturity or the experience to put the shortcomings of your teaching into perspective. Good teaching is your responsibility.

Even in these courses, be sure to use the inductive method in favor of the deductive method as a vehicle for conveying ideas (refer to Section 1.9). Any hard theorem should be suitably motivated. Do even more examples than seems necessary. Refer to the Beresford Parlett quote at the end of Section 1.9 for inspiration.

It is tempting in an upper division course to assume, at least subliminally, that your students are little mathematicians. They are not. This course may be their first exposure to rigorous thinking, to ϵ's and δ's, to Theorem–Proof–Theorem–Proof, to careful use of "for all" and "there exists", to quashing a possible theorem with a single counterexample. In short, you are not just teaching these students some advanced mathematics; you are also teaching them how to think. This is an important opportunity for you, the instructor; and it is an important juncture in the students' education. You must use it wisely.

Today's undergraduate students do not have the background and experience in rigorous thinking that we all fancy we had when we were students. They are unaccustomed to proofs and to the strict rules of logic. It is often a good idea to have a whirlwind review of logic at the beginning of an upper division mathematics course—especially in real analysis or algebra where *modus ponendo ponens*, contrapositive, proof by contradiction, induction, and so forth are used frequently. It would not be out of place to present some material on set theory and number systems as well. Some mathematics departments have a "transitions" course designed to bridge the gap in methodology between lower division courses and upper division courses. If yours is such a department, then you may moderate the advice in this paragraph.

If you do not make some extra effort to help the students in your advanced courses over the "hump" that separates math enthusiasts from mathematicians, then you are missing an opportunity to contribute to the pool of mathematical talent in this country. It's your decision, but if you decide not to participate then you have no right to complain when your department's graduate program is reduced due to lack of students, or your undergraduate program curtailed for lack of majors.

11. Time

There are several aspects of teaching that require time management skills. When you are giving a lecture, you must cover a certain amount of material in the allotted time—and at a reasonable rate. When you give a course, you must cover a certain amount of material in one semester. When you give an exam, it must be do-able by an average student in the given time slot. When you answer a question, the length of the answer should suit the occasion.

All of these topics, save the last, have been touched upon in other parts of this booklet. They require some thought, and some practice and experience, so that they become second nature to you.

Nobody can design a lecture so that the last 'QED' is being written on the blackboard just as the bell rings. There are certain precepts to follow in this regard:

- Have some extra material prepared to fill up extra time.
- If you finish your lecture with five minutes to spare, don't rocket into a new topic. You will have to repeat it all next time anyway, and students

find this practice confusing.

- If the clock shows that just five minutes remain, and you have ten or fifteen minutes of material left to present, then you will have to find a comfortable place to quit. Don't race to fit all the material into the remaining time. If possible, don't just stop abruptly, thinking that you can pick up a calculation cold in the following lecture.

  An experienced lecturer will know which will be the last example or topic in the hour, and that he/she might get caught for time. Therefore the lecturer will plan in advance for this eventuality and think of several graceful junctures at which he/she might bring the hour to a close. With enough experience, you will know intuitively how to identify the comfortable places to stop; thus end-of-the-hour time management problems can be handled on the fly. In particular, if five minutes remain then *do not begin* a ten minute example!
- If you prepare (the last part of) your lecture in units of five minutes duration, and if you are on the ball, then you should never have to run over by more than two minutes nor finish more than two minutes ahead of time. (The idea here is if there are three minutes remaining then you can include another five minute chunk without running over by more than two minutes. If there are just two minutes remaining then you should stop.)
- If you *run out of time*, do not run the class over the hour—at least not by more than a minute or two. Students have other classes to attend, and they will not be listening. If the time is gone, then just quit. Make up for your lapse in the next class (this will require some careful planning on your part). Best is to plan your lecture so that you do not fall behind.

  A special note about buzzers: Some math buildings have a loud buzzer or bell that sounds at the end of the hour. Once that buzzer sounds, all is lost. Most students will instantly start packing up their books and heading for the exit. At a school without a buzzer (especially one without a clock on the wall!) you have a bit of slack since no two wrist watches are in agreement. You may want to interpret the advice in this section according to the physical environment in which you are teaching.
- If a student asks a question that requires a long answer, don't let your answer eat up valuable class time. Tell the student that the question is ancillary to the main subject matter of the course (it had better be, or else you evidently forgot to cover an important topic) and that the question can best be treated after class. However do not let the student get the impression that the question is being given the brush off.
- Conversely, if a student asks a question for which a brief answer is appropriate (such as "shouldn't that 2 be a 3?" or "when is the next homework assignment due?") then do give a suitably brief answer. Anecdotes about your childhood in Shropshire are probably out of place.

By the way, this last is more than a frivolous remark. As we slide into our golden years, we seem to be irretrievably moved to share with our students various remembrances of things past: "it seems to me that twenty years ago students worked much harder than you people are willing to work" or "when I was a student, we put in 5 hours of study for each hour of class time" or "I used to walk six miles barefoot through the snow to attend calculus class". Trust me: students hate this. You will defeat all the other good things that you do by giving in to this temptation.

If you have the time problem under control at the level of lectures, then you will have the ability to pace your course in the large as well. You should have a good idea how much material you want to cover. And when you plan the course you should allot a certain number of lectures for each topic. If you are teaching undergraduates, then they depend on your course for learning a certain body of material (that may be prerequisite for a later course). Don't short-change them.

A test should be designed for the allotted time slot. You can rationalize giving a two hour exam in a one hour time slot by saying to yourself that there is so much material in the course that you simply *had* to make the test this long. This is nonsense. The point of the exam is not to actually test the students on every single point in the course, but to make the students *think* that they are being tested on every point in the course. Ideally, the students will study everything; but your test amounts to a spot check. Even if you had a four hour time slot in which to give the exam, you couldn't really test them on everything, now could you?

If you give a two-hour exam in a one hour time slot, then you run several risks: that students will become angry, demoralized, alienated, or all three. Telling the students not to worry about their grade of 37/100 because the average was 32/100 does not work. Students are unable to put such information into perspective.

12. Why do we Need Mathematics Teachers?

I frequently ask myself why I am necessary. Can't a student just pick up a book (or boot up a piece of software) and learn calculus or any other basic subject? Instead of charging students $10,000 to $20,000 per year to attend a university, why don't we charge them somewhat less for admission to a good library? Why don't we just sell them a box of diskettes with a computerized French course, a computerized calculus course, and so forth?

Like all topics in this book, the present one is simple and has a fairly obvious answer. But this answer needs to be articulated.

Many students read their texts with little or no understanding. They see the words but they do not understand the concepts. They need someone to tell them what is important, to give priority to the ideas, to demonstrate the techniques, to respond to their questions. This is something that a computer, or even a book, will never be able to do. When a student comes to my office to ask me a question, I can listen to the question and know at what level to pitch the answer. After I've delivered that answer I can look at the student's face and tell whether he/she has understood. A computer or a book will never be able to interact with students in this fashion.

Put a different way, the college or university mathematics teacher has (or at least ought to have) the distinction of knowing his/her entire subject. Such an instructor can put any question into context and add perspectives that we could never (at least in the foreseeable future) expect from a piece of hardware.

In a larger sense, I can make adjustments in the way that I am teaching a class depending on how well the students are absorbing the ideas. I can provide stimulating material both for the average student and for the gifted student. In short, I can interact with the students *as people,* and at their level. That is the role of the teacher; it is a role that will never be supplanted by books or computers or other inanimate objects.

It has been said that an education is the product of the interaction of two first class minds. There is truth in this, for a bright and eager student asks questions, processes the answers, and asks more. Conversely, a good teacher anticipates questions, plants the seeds of new questions, and reaps the harvest.

If you want to be an effective teacher, you should give some thought to the points being made in this section, and how you can implement them in your classroom. If you are no more effective than a book or a diskette, then you are not doing your job.

13. Math Anxiety

About fifteen years ago the concept of "Math Anxiety" was invented, probably in a school of education. We don't hear much about math anxiety in math departments because such departments are full of people who don't have it. Math anxiety is supposed to be an inability by an otherwise intelligent person to cope with quantification and, more generally, with mathematics. Classic examples of math anxiety are the successful business person who cannot calculate a tip, or the brilliant musician who cannot balance a checkbook.

I have trouble balancing my checkbook too. That is in part why I now keep track of my checking account using accounting software. But I don't think that I have math anxiety. Nor do I think I have checkbook anxiety. I'm just careless.

We could enlarge the issue and wonder whether some people have speaking anxiety or spelling anxiety. Nobody ever discusses these maladies. Does that mean that they don't exist?

What sets mathematics apart is that it is unforgiving. Most people are not talented speakers or conversationalists, but comfort themselves with the notion that at least they can get their ideas across. Many people cannot spell, but rationalize that the reader can figure out what was meant (or else they rely on a spell-checker). But if you are doing a math problem and it is not right then it is wrong. Period.

Learning elementary mathematics is about as difficult as learning to play Malegueña on the guitar. But there is terrific peer support for learning to play the guitar well. There is precious little such support for learning mathematics. If the student also has a mathematics teacher who is a dreary old poop and if the textbook is unreadable, then a comfortable cop-out is for the student to say that he has math anxiety. His friends won't challenge him on this assertion; in fact they may be empathetic.

The literature, in psychology and education journals, on math anxiety is copious. The better articles are careful to separate math anxiety from general anxiety and from "math avoidance." Some subjects who claimed to have math anxiety have been treated successfully with a combination of relaxation techniques and remedial mathematics review.

I don't think that it is healthy for a mathematics teacher to worry about math anxiety. Your job is to teach mathematics. Go do it.

14. How do Students Learn?

Psychologists, sociologists, and anthropologists have been debating this question for decades. There is no general agreement on how students learn.

It is my opinion that the very best students tend to teach themselves. The instructor points out guide posts for such a student, and then the student's native intellect takes over.

On the other hand, weak students are often quite dependent on the instructor and (perhaps) the text and the lectures. If you agree that these people are worth teaching at all, then you must be there for them. Provide good lectures and a reasonable text for them to work with. Answer their questions. Be as helpful, and as encouraging, as possible to students who have the courage to come to your office for help.

Students of middling abilities are perhaps in the majority, and they share properties with both the best students and the worst. Know with certainty that you cannot please everyone.

But also strive at least to provide some stimulation for students of all abilities. The ability to do so, without an unreasonable amount of effort on your part, can come only with experience and determination. Even if you had a class with just three students, it is likely that their levels of ability would fall into two or more disparate categories. Thus there is no escaping the realities of having an uneven audience. Being forewarned, and being thoughtful, can help you to present your course so that you teach something of value to most of the students most of the time.

Let me cast these matters in a different light. An instructor who is demanding and difficult and who omits many details from his lectures will challenge the talented students and force them to go off and learn the material on their own. Certainly that was the nature of the graduate program that I attended, and it did me a world of good. On the other hand, an instructor who is lucid and who proceeds at a comfortable pace makes everything look too easy and can lull students into a false sense of security. That instructor will also bore the gifted students. How do you address both of these phenomena in your classroom?

Because at most universities we are training a widely diverse group of students, the issues raised in the last paragraphs are unavoidable. One solution, in a fundamental course like calculus for example, is to have several tracks: a calculus course for scientists and engineers, one for pre-medical students, and one for business students. This is a commonly used method to slice up student abilities so that the spectrum in any given classroom is not so broad. Another approach to the issue of widely varying student abilities might be to rethink the traditional classroom/lecture format and divide students into groups in which they can seek their own level. The final word has not been said on how to deal with these problems. See Section 2.5 for more on the value of group work.

It is important that we continue to explore new methods to teach mathematics. It has been cogently argued that the failure rate in the large lecture system is unacceptably high; that the retention rate is unacceptably low; that America is falling behind in the technology race because we are not training our young people effectively. However, in my view it is simplistic to lay the blame for this problem on the *formal method* by which we have been teaching mathematics. No methodology is perfect.

Let us own up to the fact that many of us—especially those trained at high-powered research departments—are often not trained to care about teaching. Many of us do not. That is not the way the value/reward system is set up. The traditional methods of teaching still have much to offer, provided that they are being used by people who are properly trained and who care.

A good resource for new ideas on the teaching of mathematics is the periodical *UME Trends*. It has one or more regular columns in each issue that discuss specific teaching techniques (such as the device of mathematical POST-IT notes—see Section 2.19).

15. Computers

Computers are everywhere, especially in mathematics departments. Software for teaching mathematics is also everywhere. Most publishers of basic mathematics texts are absolutely convinced that they cannot market their product without making extensive software resources available to mathematics instructors. It is not clear, as of this writing, that much of this software is actually being used in the classroom. This is in part because most instructors are not conversant with what is available, are not comfortable with these products, or simply cannot be bothered.

Another form of software for the classroom that is being heavily touted these days is the MATHEMATICA notebook. Briefly, these are self-contained environments by means of which students can interact with the computer over various mathematical issues *without* knowing anything about computing or computer languages.

All of this hardware and software raises fundamental issues about the way that mathematics is taught and the way that it ought to be learned. I drive my car every day and am perfectly comfortable in doing so while not knowing in intimate detail how it works. Ditto for my computer, my telephone, my television, and so forth. Some would say that a measure of how civilized we are is how many black boxes we are willing to use unquestioningly. To what extent should this point of view be allowed in the mathematics classroom?

Most of us were trained with the idea that the whole point of mathematics is to understand precisely why things work. To make the point more strongly, this attitude is what sets us apart from laboratory scientists. We make no statement unless we can prove it. We use no technique unless we fully grasp its inner workings. We would not dream of using the quadratic formula unless we knew where it came from. We would not be comfortable using the fundamental theorem of calculus unless we had seen its proof.

Yet in many colleges these days the linear programming course consists primarily of learning to use LINDO or any number of other canned packages for applying the simplex method and its variants. The statistics course consists primarily in learning to use SAS. The undergraduate numerical analysis course consists in learning to use IMSL.

It has been argued that the use of programmable calculators and MATHEMATICA notebooks in the calculus classroom will free students from the drudgery of calculation and will allow us to teach them how to analyze multi-step word problems (see, for instance, [STE]). Thus, it is hoped, our lower division calculus classes will be more closely tied to the way that mathematics is used in the real world. I see this as double-talk.

It is certainly the case that the less able students are so hampered by their inability to take the derivative and set it equal to zero or to find the roots of a quadratic equation or to calculate the partial fraction decomposition of a rational expression that they have little hope of successfully analyzing a multi-step word problem whose solution includes one or more of these techniques. But there is no evidence to support the (apparent) contention that a person who is unable to use the quadratic formula will somehow, if these technical difficulties are handled for him/her by a machine, be able instead to analyze conceptual problems. Using the quadratic formula is easy. Analyzing word problems is hard. A person who cannot do the first will also probably not be able to do the second—with or without the aid of a machine.

I think, as I have already indicated, that the computer can be used successfully to provide helpful graphical analyses. If you want to draw a surface in space and then move it around to analyze it from all sides, then there is nothing to beat MATHEMATICA. If you want to illustrate Newton's method, or use the Runge-Kutta method, or implement Simpson's rule, then a computer is almost indispensable. But the use of the computer should be based on a firm foundation of conceptual and technical understanding.

Some of the federally funded calculus projects have been teaching the entire calculus sequence from MATHEMATICA notebooks. The principal investigators in these projects claim that they can take students with no particular interest in mathematics, who have traditionally done poorly in their mathematics courses, and ignite in them a spark for the subject using these notebooks. They claim that the students get truly excited from interacting with the machine. And students who have never before received a good math grade before end up getting an "A" in the course. However follow-ups have been done on these students and it has been determined that the majority of them have no comprehension of the subject matter and little retention. When confronted with this information, one principal investigator has said "I think that the concept of 'understanding' has traditionally been over-emphasized."

What does this *mean*? If we are not trying to endow our students with understanding then just what are we doing? I think that it is perfectly all right for a trade school or a commercial business college to teach linear programming by training its students to use LINDO. But in a university mathematics department a student of linear programming should be taught the simplex method and why it works.

It is probably the case that students learning calculus from MATHEMATICA notebooks respond positively to the extra attention they are receiving, to the novelty of the teaching environment, and to the fact that making a mistake when interacting with the computer is less heinous than asking a dumb question in class. In the MATHEMATICA notebook environment, the student is probably more willing to try things and to experiment. We should think carefully about how to capitalize on the special features that computers can contribute to the learning experience. However it is clear that we do not yet know all the answers, nor have we realized all the potential.

Consider this: in my view, a student is better off spending an hour with a pencil—graphing functions just as you and I learned—than generating fifty graphs on a computer screen in the same time period. From first hand experience, I am absolutely sure what the first exercise will teach the student. It is not at all clear what the student gains from the second.

A wise man once told me that the computer is a solution looking for a problem. It is obviously a powerful tool in the right circumstances. The generation of minimal surfaces of arbitrary genus by Hoffman, Hoffman, and Meeks ([HOF]) is a dazzling use of computers; the simulations that these three mathematicians performed would have been inconceivable without this advanced computer graphics technology. MATHEMATICA is a powerful tool in the hands of the right user, as are AXIOM, MACSYMA and MAPLE. However it is not apparent that we have yet found the right use for the computer in the calculus classroom. There is no data to support the contention that the use of computers, or even calculators, in the classroom leads to a more effective *long term* learning experience.

Calculus is perhaps the most powerful body of analytic tools ever devised. All young scientists should learn calculus in essentially the traditional fashion so that they have these tools at their disposal. *Graphing a function* is one of the most basic processes of analytical thinking—analogous to finger exercises for the piano. The partial fractions technique is also one of the most far-reaching algebraic devices that we have. Integration by parts is perhaps the most ubiquitous and powerful tool in all of mathematical analysis. Letting a computer do these processes for the students abrogates much of what we have learned in the last three hundred years.

Let me assure you that I have discussed this point with the presidents of large high tech corporations and they agree with me absolutely. Use of the new technology should be layered atop a traditional foundation. That is what works in the classroom and that is what works in the real world.

16. Applications

One of the most chilling things that can happen to an unprepared, unseasoned faculty member is to have a belligerent student raise his/her hand and say "What is all this stuff good for?" And one of the most irresponsible things that a faculty member can say in response is "I don't know. That is not my problem." If you do not have an answer for this student question then you are not doing your job.

I have found it useful in all of my undergraduate classes to tell the students about applications of the techniques being presented *before* the aforesaid chilling question ever comes up. This requires a little imagination. If I am lecturing about matrix theory then I tell the students a little bit about image processing and image compression. Or I tell them about eigenvalue asymptotics for clamped beams and applications to the building of a space station (not coincidentally, this is a problem that I have worked on in my research). If I am lecturing about surfaces then I tell them about the many applications of surface design problems. If I am lecturing about uniform continuity or uniform convergence then I tell them about some of the applications of Fourier analysis. The pedagogical technique that I am describing in effect defuses any potential belligerence from engineering or other students who have no patience for mathematical abstraction.

Carrying out this teaching technique requires a little forethought and a little practice. After a while it becomes second nature, and you will find yourself thinking of potential answers while taking a shower or walking to class. If it suits your style, keep a file of clever applications of elementary mathematics. It is not true that the concept of uniform convergence is used on a daily basis by engineers to construct bridges. Do *not* use this facile line of reasoning to talk yourself into abandoning the effort to acquaint lower division students with the applications of mathematics. Instead, reason that uniform convergence is a bulwark of the theory behind the practical applications of mathematics. It is important. Act as though you believe it.

Try to be flexible and to reach out for up-to-date and striking applications. Uniform convergence is a basic idea in the convergence of series; one of the most interesting uses of series is in Fourier analysis. And what is Fourier analysis good for? Mention the hot new theory of wavelets and some of its uses. Go from there.

It is a matter of personal taste (and much debate) as to how much should be done with applications in, say, a calculus class. For most of us the problem is solved by the very nature of the undergraduate curriculum. There is little time to do any but the most routine applications. But times are changing and many mathematics departments are re-evaluating their curricula. There is considerable enthusiasm for infusing the freshman-sophomore curriculum with more applied material. Examples of a refreshing new approach to the calculus, by way of applications, can be seen in the Amherst project materials [AMH] and the Harvard materials [HAL].

If you decide to work applications into your class then consider this. Mathematical modeling is complicated and difficult. If you take an already complex mathematical idea and spend an hour applying it to analyze a predator-prey problem, or to derive Kepler's third law, or to design the Wankel engine, then you are likely to lose all but the most capable students in the room. How will you test them on this material? Can you ask the students to do homework problems based on this presentation?

Think of what it is like to teach the divergence theorem. There are almost too many ideas, layered one atop the other, for a freshman or sophomore to handle. Students must simultaneously keep in mind the ideas of gradient, surface, surface integral, curl, and so forth. Most cannot do it. The same phenomenon occurs when one is attempting to get students to understand a really meaty application. I am not advising you against doing these applications. What I *am* saying is this: if you should choose to do one, go into it with your eyes open. If, after fifteen minutes, the students' eyes glaze over then you will have to shift gears. Be prepared with a physical experiment, or a film strip, or an overhead slide, or a transition into another topic.

An effective presentation of an application should be broken into segments: a little analysis, a little calculation, a little demonstration. Lower division students cannot follow a one hour analysis. Always bear in mind that, no matter how satisfying you find a particular application, your audience of freshmen may be somewhat less enthusiastic and may require help and encouragement.

On the other side of the coin, don't get sucked into doing just trivial, artificial applications. This cheapens our mission in the students' eyes and makes us seem less than genuine. The calculus and its applications are among the great achievements of western civilization. Be proud to share with the class the analytical power of calculus. Do so by presenting some profound applications, but put some effort into making the presentation palatable.

My experience is that, for freshmen, short applications are the best. They can be modern, they can be interesting, but if doing the analysis entails layering too many levels of ideas on top of each other then most students will be lost. And there is always the danger that some student will ask the question "will this be on the test?" What can you say? Will an hour-long application be on the test? No, but some of the analytical techniques that you use in the example could be. You had better have an answer prepared for questions such as these. See also Section 3.5 on answering difficult questions.

William Thurston, in his article [THU] on the teaching of mathematics, points out that mathematics is a "tall" subject and that mathematics is a "wide" subject. The tallness articulates the fact that mathematics builds up and up, each new topic taking advantage of previous ones. It is wide in the sense that it is a highly diverse and interactive melange. It interfaces with all of the other sciences, with engineering, and with psychology and many other disciplines. It is our job as teachers of mathematics to introduce students to this exciting field, and to motivate students to want to study mathematics and to major in it. Applications are a device for achieving this end. Using them wisely and well in the classroom is a non-trivial matter. Talk to experienced faculty in your department about what resources are available to help you present meaningful applications to your classes.

CHAPTER II

Practical Matters

1. Voice

There is nothing more stultifying than a lecture in a reasonably large class on a hot day delivered by an oblivious professor mumbling to himself at the front of the room. We are not all actors or comedians or even great public speakers. But we are teachers, and we must convey a body of material. We must capture the class's attention. We must *fill the room.*

I am not saying that you must lose your dignity, or act silly, or show off. You must learn to use your voice and your eyes and your body and your presence as a tool. If you are going to say something important, then make a meaningful pause beforehand. *Say* that it is important. Repeat the point. Write it down. Give an example. Repeat it again.

You can gain the attention of a large group by lowering your voice. Or by raising it. Or by pausing. One thing is certain: you will not gain the audience's attention by rolling along in an uninflected monotone. Again, I am not suggesting that you undergo a personality change in order to be an effective teacher.

At a well-known university in southern California they once tried bringing in actors from Hollywood to help professors spice up their delivery. Such pandering is inappropriate, offensive, and childish.

What I *am* suggesting here is that you take just a little time and contemplate your lecture style. A lecture or class should be a controlled conversation with your audience. It is a trifle one-sided, of course. But there must be cerebral interaction between the teacher and the students. That means that you, the instructor, must grab and maintain the attention of the class. Your behavior in front of the group is a primary tool for keeping the lines of communication open.

When you are talking about a subject that you perceive to be trivial, and when you are nervous, you tend to talk too fast. Novice instructors find themselves barreling through their lectures. You must resist this tendency. If you are really new at the business of teaching, then practice your lectures. Get a friend to listen. In calculus, a fifty minute lecture with four or five good examples and some intermediate explanatory material is probably just about right (I'm thinking here of a lecture on max-min problems, for example). Try to make each lecture consist of about that much material, and make it fill the hour. If you finish early, that is fine (but it may mean that you talked too fast). You can quit early for that day, or do an extra example, or use the extra time to answer questions.

Don't give the students the impression that you are in a rush. It puts them off, and reflects a bad attitude toward the teaching process. If on Wednesday you plan to explain the chain rule, then do just that. If the chosen topic does not fill the hour, then do an extra example or field questions. Do not race on to the next topic. One idea per lecture, at the lower division level, is about right. [Of course if you are teaching a multi-section class at a big university, then it is important to keep pace with the other instructors. This is yet another reason for keeping careful track of your use of time. See also Section 1.11.]

It is something of an oversimplification, but still true, that a portion of the teacher's role is as a cheer leader. You are, by example, trying to convince the students that this ostensibly difficult material is do-able. Part of the secret to success in this process is to have a controlled, relaxed voice, to appear to be at ease, and to be organized. Don't let a small error fluster you. Make it seem as though such a slip can happen to anyone, and that fixing it is akin to tying your shoelace or pulling up your socks.

But, as with all advice in this booklet, you must temper the thoughts in the last paragraph with a dose of realism. If you make the material look very easy, then students will infer that it *is* very easy. The psychological processes at play here are not completely straightforward. Nobody would be foolish enough to go to an Isaac Stern concert and come away with the impression that playing the violin is trivial. Yet students attend my calculus lectures, watch me solve problems, conclude that the material is easy and that they have it down cold, decide that in fact they *don't* need to do any homework problems or read the book, and then flunk the midterm.

These are the same students who come to me after the exam and say "I understand all the ideas. The material is absolutely clear when you talk about it in class. But I couldn't do the problems on the exam." I like to tease my students by reminding them that this is like saying "I really understand how to swim, but every time I get in the water I drown."

On the one hand, you don't want to make straightforward material look hard. After 300 years, we've got calculus sewn up. There is no topic in the course that is intrinsically difficult. We merely need to train our students to do it. So *do* make each technique look straightforward. But remind the students that *they themselves need to practice*. Do this by telling them so, by giving quizzes, by varying the examples and introducing little surprises. Ask the class questions to make the students turn the ideas over in their own minds. Use your *voice* to encourage, to wheedle, to cajole, to question, to stimulate.

Even if you know how to use your voice effectively with a small audience, there are special problems with the large audiences that occur in the teaching of calculus (for instance) at many universities. Refer to Section 2.13 for more on this matter.

2. Eye Contact

We all know certain people who invariably emerge as the leader of any group conversation. Such people seem to sparkle with wit, erudition, and presence. They have a sense of humor, and they are intelligent. What is their secret?

The answer is multi-faceted. This is obviously a talent that you must cultivate. Part of the trick is to show genuine interest in what other people have to say before bounding ahead with what you have to say. Another part is to talk about subjects, and to tell anecdotes, that you know will interest other people. Nothing is more boring, for instance, than listening to a half hour discourse on collecting firewood in the forest if you yourself are not disposed to this activity.

Many of the devices that make for an engaging conversationalist also make for an engaging lecturer. A review of the last paragraphs, and the rest of this booklet, will bear out this assertion. The device that I want to dwell on here is *eye contact.*

Telling a good joke while staring at the floor with your thumb in your ear will not have the same effect as telling the joke while looking at your listener, engaging his/her attention, and reacting to the listener while the listener is reacting to you. A good joke teller has his audience starting to chuckle half way through the joke and just dying for the punch line. Getting a good laugh is then a foregone conclusion.

Giving a good lecture is serious business, and is not the same as telling a joke. But many of the moves are the same. If you want to hold your audience's attention then you must look at your audience. You must engage not one individual but all. A good lecturer speaks to individuals in the audience, to grouplets in the audience, and to the whole audience. Like a movie camera, you must zoom in and zoom out to get the effects that you wish to achieve. A ninety minute movie filmed at the same constant focal length would be dreadfully boring. Ditto for a lecture.

Some people are very shy about establishing eye contact. It is a device that you must consciously cultivate. The end result is worth it: the lecturer who can establish eye contact is also the lecturer who is confident, who is well prepared, and who delivers a good lecture.

3. Blackboard Technique

Write neatly. Write either in very plain long hand or print. [Some people object to printing because of the clickety-click of the chalk. You'll have to make your own peace with this objection.] Be sure that your handwriting is large enough. Be sure that it is dark enough. Endeavor to write straight across the blackboard in a horizontal line. Proceed in a linear fashion; don't have a lot of insertions, arrows, and diagonally written asides.

Don't put too much material on each board. The ideas stand out more vividly if they are not hemmed in by a lot of adjacent material. In particular, it is difficult for students to pay attention when the teacher fills the board with long line after long line of neat print. An excellent guitarist once said that the silences in his music were at least as important as the notes. When you are laying material out on a blackboard, the same can be said of the blank spaces.

Label your equations so that you can refer to them verbally. Draw sketches neatly. Use horizontal lines to set off related bodies of material.

You can control your output more effectively by keeping the length of each line that you write short. Think of the blackboard as being divided into several boxes and write your lecture by putting one idea in each box. If necessary, *actually divide the blackboard into boxes.*

If the lecture hall has sliding blackboards, think ahead about how to use them so that the most (and most recent) material is visible at one time. For those combinatorial theorists among you, or those experts on the game of NIM, this should be fun.

If you are right-handed, consider starting at the right hand extreme of the blackboard space and working left. The reason? That way you are never standing in front of what you've written. Good teaching consists in large part of a lot of little details like this. You shouldn't be pathological about these details, but if you are aware that they are there then you will pick up on them.

Try to think ahead. Material that needs to be kept should be written on a blackboard to the far left or right where it is out of the way but can be referred to easily. You may wish to reserve a box on the blackboard for asides or remarks. This is another aspect of the precept that you know the material cold so that you can concentrate on your delivery. Just as an actor knows his lines cold so that he can make bold entrances and exits, and not trip over his feet, so you must be able to focus a significant portion of your brain on the *conveying of the information.*

If your lecture will involve one or more difficult figures then practice them on a sheet of paper in advance. Remember that you are a mathematical role model for the students. If you make it appear that it is difficult for *you* to draw a hyperboloid of one sheet, then how are the students supposed to be able to do it? Of course you can prepare the figure in advance on an overhead slide. This solves the problem of having a nice figure to show the students; it does not solve the problem of *showing* the students how to draw the figure. If necessary, consult a colleague who is artistically adept for tips on how to draw difficult figures.

If you cannot organize the steps of a maximum-minimum problem, then can you really expect the students to do so? In the best of all possible worlds, the students' work is but a pale shadow of your own. So your work should be the platonic ideal. Sometimes, in presenting an example or solving a problem, you may inadvertently gloss from one step to another; or you might make a straightforward presentation look like a bag of tricks. This is very confusing for students, especially the ones who lack confidence. By organizing the solution in a step-by-step format you can avoid these slips.

After you have filled a board, it should be neat enough and clear enough that you could snap a polaroid shot and read the lecture from the polaroid. In particular, you should not lecture by writing a few words, erasing those, and then writing some more words on top of the erased old words. Students cannot follow such a presentation. I cannot emphasize this point too strongly: Write from left to right and from top to bottom. *Do not erase.* When the first box is filled, proceed to the second. *Do not erase.* Only when all blackboards are full should you go back and begin erasing. Students must be given time to stare at what they've just seen as well as what is currently being written. Keep as much material as possible visible at all times.

Do not stand in front of what you are writing. Either stretch out your arm and write to the side or step aside frequently. Read what you are writing to the class. Make the mathematics happen before their eyes and *be sure that they can see everything. Say the words as you write them.* Every once in a while, pause and step aside to catch your breath and to let them catch up.

Here is a common error that is made even by the most seasoned professionals. Imagine that you do an example that begins with the sentences "Find the local maxima and minima of the function ... " And so forth. Say that you've worked the example. Now suppose that the next example begins with the same phrase. It is a dreadful mistake to erase all but that first phrase and begin the new example on the fly, as it were.

Why is this a mistake?—it *seems* perfectly logical. But the students are taking notes! How can they keep up if you pull a stunt like this? *Slow yourself down.* Write the words again. If a student gets two sentences behind then he/she may as well be two paragraphs behind. Give frequent respites for catch up.

And now a coda: how much of what you are saying should you write? In my experience, the answer is "As much as possible." When you are transmitting sophisticated technical ideas verbally, students have trouble keeping up. Many of them are not native English speakers. They need a little help. I write down everything except asides. I say the words as I write them. This is also a device for slowing myself down. Most of us tend to talk far too fast—at least about mathematics. Because of my poor handwriting, I *must* write deliberately when I lecture and this serves as an ideal counterpoint to my otherwise rapid speech. Each individual instructor will have to decide for himself how to strike a balance here.

There is a real psychological barrier for the instructor to overcome when learning blackboard technique, and voice control. When we understand very deeply what we are talking about, then it all seems quite trivial. We can convince ourselves rather easily—at least at a subconscious level—that it is embarrassing to stand in front of a group and enunciate whatever mundane material is the topic of the day. Thus we are inclined to race through it, both verbally and in the way that we render it on the blackboard. *Be conscious of this trap and do not fall into it.* I have never been criticized for being too clear, whether I was giving a calculus lecture or a colloquium lecture. *Slow down.* Be deliberate. Enunciate. Explain.

A mistake that we all make is to talk too much and write too little. Especially at the start of the hour, we tend to hyper-verbalize. Remember that this is technical material, the acoustics are not perfect, and many of the students are not native speakers. Writing will help you to slow yourself down and will help students to keep up. This is just a way of making yourself clear. Material that is rattled off verbally, and not written, appears to be unimportant. If it is important, write it down and *write it down clearly.* You need not write "Good morning class. Today is Wednesday." But write at least the key technical ideas that you are discussing.

4. Body Language

If you skulk into your classroom, stand slouching in front of the class with a furtive and disreputable expression, and are wearing slovenly clothing to boot, then you are sending numerous negative signals to your class. It sounds trite to say it, but dress neatly and attractively when you go to teach. Stand erect and look dignified. Attending to these mundane matters really does make a difference.

When you are teaching an elementary class, there is a tendency to suppose that everything you are saying is trivial. The upshot is that you talk too fast and write too fast *or not at all.* (See also Section 2.3 on Blackboard Technique.) Especially at the beginning of the hour, or at the start of a new topic, there is a great temptation to just rattle on (verbally) while writing little or nothing. This is a big mistake.

When you are teaching technical material to freshmen, it is impossible to be too clear. Write *everything* down. Write it neatly. Write it slowly. If a young student is six words behind then he/she may as well be six paragraphs behind. If you say "Recall that the interior extrema of a continuously differentiable function occur at points where the derivative is zero" then you have already lost most of them. Really. Write it down, at least in abbreviated form.

It is also the case that writing it down neatly and slowly is a subliminal way of telling the students that this material is important. If you are taking the trouble to write it down deliberately, then it must be worth writing deliberately. Conversely, if you scribble some incoherent gibberish, or scribble nothing at all, then what signal are you sending to the students?

5. Homework

In most lower division courses, and many upper division ones, it is by way of the homework that you have the greatest direct interaction with your students. When students waylay you after class or come to your office hour, it is usually to ask you about a homework problem. This is why the exercise sets in a textbook are often the most important part of the book (textbook authors do not seem to have caught on to this observation yet) *and* why it is critical that homework assignments be sensibly constructed.

Let me stress again that I am not trying to sell a time-consuming attitude or habit to you. If you take twenty minutes to compose a homework assignment then you are probably taking too much time. But consider the following precepts:

- Do not make the homework assignment too long.
- Do not make the homework assignment too short.
- Be sure that the assignment touches on all of the most important topics.
- Be sure that the homework assignment drills the students on the material that you want them to learn and the material that you will be testing them on.
- *Make sure that at least some of the homework problems are graded.*
- Plan ahead: the exams that you give should be based only on material that the students have seen in the homework.

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