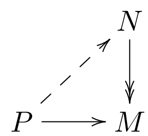
ALGEBRAIC K-THEORY AND QUADRATIC RECIPROCITY

Much of this comes right out of Milnor's delightful "Introduction to Algebraic K-Theory"

1. Step 0

1.1. What is K_0?

- What are projective modules?
  - P is projective if there exists Q so that $P \oplus Q \simeq \Lambda^n$
  - we get the following diagram:



- K_0 as classes of projective modules where $[P] + [Q] = [P \oplus Q]$
  - $K_0(F) \simeq \mathbb{Z}$
  - $K_0(\mathbb{Z}) \simeq \mathbb{Z}$ (true for all PIDs, local rings)
- Ring structure through tensor product: $[P \otimes Q] = [P] \cdot [Q]$
- Functoriality
  - $i : \mathbb{Z} \to \Lambda$ gives $i_* : K_0(\mathbb{Z}) \to K_0(\Lambda)$
  - if $j : \Lambda \to F$, then $j_* : K_0(\Lambda) \to K_0(F)$
  - $K_0(\Lambda) \simeq \mathbb{Z} \oplus \tilde{K}_0(\Lambda)$
  - $K_0(\Lambda) \simeq \mathbb{Z} \oplus C(\Lambda)$ when Λ is a Dedikind domain

2. Step 1

2.1. What is K_1?

- Constructing $GL(\Lambda)$ as limit of $GL_n(\Lambda)$
- Elementary matrices as group (multiplicatively) generated by "$kR_i + R_j$" row operations
- $K_1(\Lambda) \simeq GL(\Lambda)/E(\Lambda)$
- $E(\Lambda) = [GL(\Lambda), GL(\Lambda)]$

  $K_1(\Lambda)$ as abelianization of $GL(\Lambda)$
- There is natural map $K_0(\Lambda) \otimes K_1(\Lambda) \to K_1(\Lambda)$ when Λ is commutative
- Congruence subgroup problem for $\mathcal{O}$ a ring of algebraic integers of number field
  - Let $\Gamma_{\mathfrak{q}} = \ker(SL_n(\mathcal{O}) \to SL_n(\mathcal{O}/\mathfrak{q}))$
  - If $\Gamma_{\mathfrak{q}}$ is in some subgroup of $SL_n(\mathcal{O})$, then that subgroup is finite index
  - Are there finite subgroups of $SL_n(\mathcal{O})$ which don't contain some $\Gamma_{\mathfrak{q}}$?

3. Step 2

3.1. What is K_2?

- Let e_{ij}^{λ} be the matrix with λ in the i,j-position, and 1's along the diagonal
- Elementary matrices e_{ij}^{λ} and e_{kl}^{μ} satisfy the following relation:

$$[e_{ij}^{\lambda}, e_{kl}^{\mu}] = \begin{cases} 1, & \text{if } j \neq k, i \neq l \\ e_{il}^{\lambda\mu}, & \text{if } j = k, i \neq l \\ e_{kj}^{-\mu\lambda}, & \text{if } j \neq k, i = l \end{cases}$$

- $K_2(\Lambda)$ is all relations amongst elementary matrices modulo these "obvious" ones
- Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $(e_{12}^{1} e_{21}^{-1} e_{12}^{1})^4 = I$

3.2. A simpler description for fields.

- $K_2(F) = F^{\times} \otimes F^{\times}$ modulo the ideal generated by $\{f \otimes (1-f) : f \in F^{\times} \setminus \{1\}\}$.
- Steinberg symbols are bimultiplicative $c : F^{\times} \times F^{\times} \to A$ so that $c(x, 1-x) = 0$
- $K_2(F)$ is the "universal Steinberg symbol"
- $\{x, -x\} = 1$ and also $\{x, y\} = \{y^{-1}, x\}$
  - $\{x, -x\} = \{x, \frac{1-x}{1-x^{-1}}\}$
  - $\{x, y\} = \{x, y\}\{x, -x\}\{xy, -xy\}^{-1}\{y^{-1}, -y^{-1}\}$
- if v is a discrete valuation on F, Λ the valuation ring and $\mathfrak{P}$ the maximal ideal, then

$$d_v(x, y) = (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \mod \mathfrak{P}$$

  is a Steinberg symbol with values in $\bar{F}^{\times} = (\Lambda/\mathfrak{P})^{\times}$

3.3. Why do number theorists care? (Part I).

- For p an odd prime, let $(x, y)_p \in \mathbb{F}_p^{\times}$ be the Steinberg symbol from p-adic valuation
  - note: if $(p, x) = 1$, then $(x, p)_p = x \mod p$.
- For $p = 2$, we need a new symbol definition since $\mathbb{F}_2^{\times}$ is stupid
  - write $x = 2^{j(x)} x' \in \mathbb{Q}$, and then

$$[i(x), k(x)] = \begin{cases} [0,0], & \text{if } x' \equiv 1 \mod 8 \\ [1,1], & \text{if } x' \equiv 3 \mod 8 \\ [0,1], & \text{if } x' \equiv 5 \mod 8 \\ [1,0], & \text{if } x' \equiv 7 \mod 8 \end{cases}$$

  - define $(x, y)_2 = (-1)^{i(x)i(y) + j(x)k(y) + k(x)j(y)}$
  - note $(p, q)_2 = \begin{cases} 1, & \text{if either } p \equiv 1 \mod 4 \text{ or } q \equiv 1 \mod 4 \\ -1, & \text{if } p \equiv q \equiv 3 \mod 4 \end{cases}$
- It turns out that $K_2(\mathbb{Q}) \simeq \mathbb{F}_2 \oplus \mathbb{F}_3^{\times} \oplus \mathbb{F}_5^{\times} \oplus \cdots$ by the map $\{x, y\} \mapsto (x, y)_2 \oplus (x, y)_3 \oplus (x, y)_5 \oplus \cdots$

Proof. Let L_m be the subgroup of $K_2(\mathbb{Q})$ generated by symbols $\{x, y\}$ where $|x|, |y| \leq m$. Note $L_{m-1} = L_m$ unless m is prime.

We'll show that $L_p = \mathbb{F}_2 \oplus \mathbb{F}_3^\times \oplus \cdots \oplus \mathbb{F}_p^\times$ by showing that $L_p/L_{p-1} \simeq \mathbb{F}_p^\times$ (and using induction). Notice that

$$L_2 = \{\{-1, 1\}, \{1, 1\}, \{1, -1\}, \{-1, -1\}\} = \{\mathrm{id}, \{-1, -1\}\} \simeq \mathbb{F}_2.$$

(This is our base case.)

We define the map $\Phi : \mathbb{F}_p^\times \to L_p/L_{p-1}$ by $\Phi(x) = \{x, p\}$.

– well-defined?

– surjective? L_p is generated by $L_{p-1}, \{p, x\}, \{x, p\}, \{p, p\}$. Note

$$\mathrm{id} = \{-p, p\} = \{-1, p\}\{p, p\}$$

□

- Universality says that for any $c : \mathbb{Q}^\times \times \mathbb{Q}^\times \to A$ there exist $\phi_p : \mathbb{F}_p^\times \to A$ (and $\phi_2 : \mathbb{F}_2 \to A$) so that $c(x, y) = \prod_p \phi_p((x, y)_p)$
- When $(x, y)_\infty = \left\{ \begin{array}{ll} 1, & \text{if } x > 0 \text{ or } y > 0 \\ -1, & \text{if } x, y < 0 \end{array} \right\}$, it turns out that we get

$$(x, y)_\infty = (x, y)_2 \prod_p (x, y)_p^{\frac{p-1}{2}}$$

Proof. Certainly $\phi_p((x, y)_p) = \left((x, y)_p^{\frac{p-1}{2}} \right)^{\epsilon_p}$ where ϵ_p is 0 or 1.

– Take $x = y = -1$. Then

$$-1 = (-1, 1)_\infty = (-1, -1)_2^{\epsilon_2}.$$

– If $p = 8k \pm 3$ then use $x = 2$, $y = p$:

$$1 = (2, p)_\infty = (2, p)_2 \left((2, p)_p^{\frac{p-1}{2}} \right)^{\epsilon_p}$$

– If $p = 8k + 7$ then use $x = -1$, $y = p$

– If $p = 8k + 1$, then there exists a prime $q < p$ so that p isn't a residue mod q. Now plug in $x = q, y = p$:

$$1 = (p, q)_\infty = (p, q)_2 \left((p, q)_q^{\frac{q-1}{2}} \right)^{\epsilon_q} \left((p, q)_p^{\frac{p-1}{2}} \right)^{\epsilon_p} = -\left((p, q)_p^{\frac{p-1}{2}} \right)^{\epsilon_p}$$

□

- Plug in p, q and you get quadratic reciprocity