Entropy, Strings, and Partitions of Integers

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1. Preliminaries

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Example. $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

There's a formula for $p(N)$:

$$p(N) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(N)\sqrt{k}\frac{d}{dN}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(N-\frac{1}{24}\right)}\right)}{\sqrt{N-\frac{1}{24}}}\right),$$

where

$$A_k(N) = \sum_{0\le m<k\,,\,(m,k)=1} e^{\pi i(s(m,k)-2Nm/k)}$$

and

$$s(m,k) = \sum_{j=1}^{k-1} \frac{j}{k}\left(\frac{jm}{k} - \left\lfloor\frac{jm}{k}\right\rfloor - \frac{1}{2}\right).$$

(Rademacher 1937)

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$$\ln p(N) \sim 2\pi\sqrt{\frac{N}{6}}\,.$$

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Proof. Expand the right-hand side:

$$\underbrace{(1 + x^1 + (x^1)^2 + \cdots)}_{\ell=1}\,\underbrace{(1 + x^2 + (x^2)^2 + \cdots)}_{\ell=2}\,\underbrace{(1 + x^3 + (x^3)^2 \cdots)}_{\ell=3}\cdots\,.$$

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Therefore, the coefficient of x^N in the product is the number of partitions of N.

2. The Quantum String

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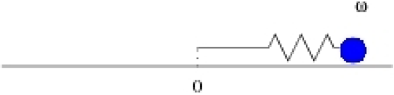
A **harmonic oscillator** is a system that vibrates back and forth at some frequency ω, such as a mass on a spring.



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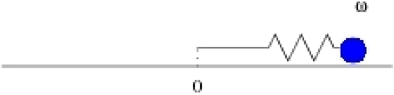

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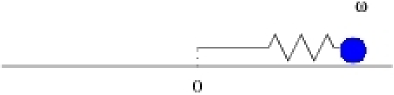

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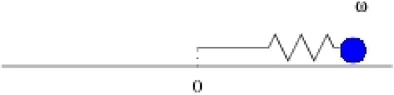

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Terminology: An oscillator with energy E_n "is in state n" or "has n excitations at frequency ω."

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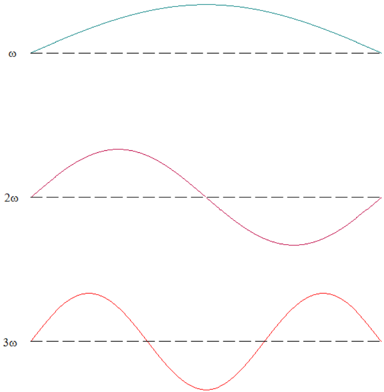

The string's allowed frequencies of vibration are positive integer multiples of this fundamental frequency:

$$\omega, 2\omega, 3\omega, \ldots .$$

So we can think of a string as being made up of an infinite collection of harmonic oscillators, with frequencies $\omega, 2\omega, 3\omega, \ldots$.

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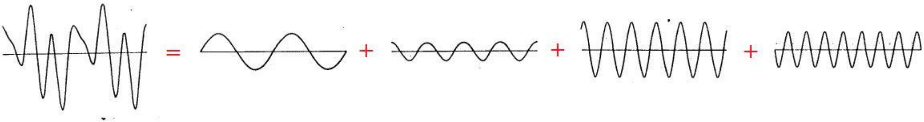
Any pattern of vibration can be decomposed as a linear combination of these basic modes of vibration,

$$y(x) = a_1 \sin \omega x + a_2 \sin 2\omega x + a_3 \sin 3\omega x + \cdots .$$



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Therefore, the state of a quantum string is specified by a sequence ψ of non-negative integers,

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The total energy of a string in state ψ is the sum of the energies in each of the individual oscillators,

$$\boxed{E = n_1\hbar\omega + n_2\hbar(2\omega) + n_3\hbar(3\omega) + \cdots \equiv N\hbar\omega}\,,$$

where

$$\boxed{N = n_1 + 2n_2 + 3n_3 + \cdots}\,.$$

There is a correspondence between states ψ of the string and partitions of $N = n_1 + 2n_2 + 3n_3 + \cdots$:

The state $\psi = (n_1, n_2, n_3, \dots)$ corresponds to the partition

$$N = \overbrace{1 + 1 + \cdots + 1}^{n_1 \text{ summands}} + \overbrace{2 + 2 + \cdots + 2}^{n_2 \text{ summands}} + \overbrace{3 + 3 + \cdots + 3}^{n_3 \text{ summands}} + \cdots .$$

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So if we can count how many string states there are with energy $N\hbar\omega$, we'll have solved our partition problem!

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These features depend only on the aggregate properties of the particles, so we can use statistical methods to study them.

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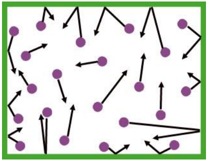

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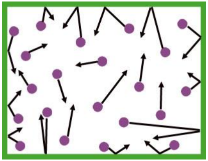

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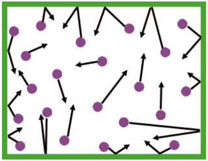

There are many microstates—call their number W—that give rise to this macrostate.

The **entropy** of the system is defined as

$$\boxed{S = k \ln W},$$

where $k = 1.38 \times 10^{-23}$ J/K is **Boltzmann's constant**.


S = k. log W
LVDWIG
BOLTZMANN
1844-1906
DR PHIL. PAULA
BOLTZMANN
GEB. CHIARI
ARTHUR
BOLTZMANN
LVDWIG
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Note. For a system with W equally probable microstates, $p_\psi = 1/W$ for all ψ, so that

$$S = -k \sum_{\psi=1}^{W} \frac{1}{W} \ln\left(\frac{1}{W}\right) = -kW\left(\frac{1}{W}\right)(-\ln W) = k \ln W\,.$$

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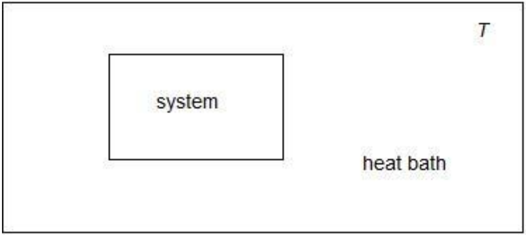

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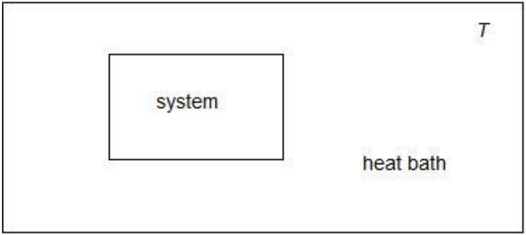

What is the probability p_ψ of finding our system in microstate ψ with energy E_ψ?

Imagine an **ensemble** consisting of many copies of our system in contact with the heat bath.

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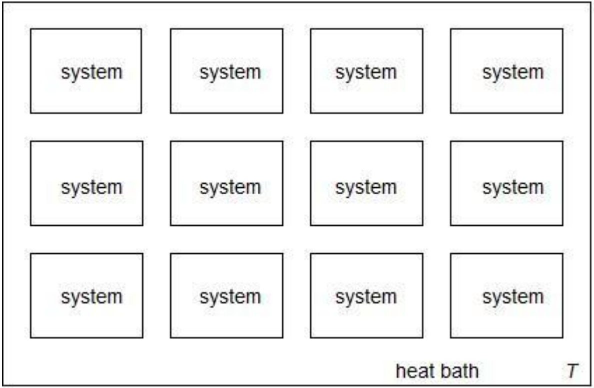

Let	a_{ψ_1}	be the number of copies in state	ψ_1	with energy	E_{ψ_1},
	a_{ψ_2}		ψ_2		E_{ψ_2},
	a_{ψ_3}		ψ_3		E_{ψ_3}.

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Since $$\frac{a_{\psi_3}}{a_{\psi_1}} = \frac{a_{\psi_2}}{a_{\psi_1}} \cdot \frac{a_{\psi_3}}{a_{\psi_2}},$$

$$f(E_{\psi_3} - E_{\psi_1}) = f(E_{\psi_2} - E_{\psi_1}) f(E_{\psi_3} - E_{\psi_2}).$$

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for some constant β, and likewise

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This probability distribution is called the **Boltzmann distribution**.

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$$p_\psi = \frac{1}{Z} \exp\left(-\frac{E_\psi}{kT}\right) ,$$

we can determine Z by noting that our system must be in *some* microstate, so

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The factor Z is called the **partition function**. It enables us to compute any macroscopic property of our system.

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We can now compute various physically important macroscopic quantities, starting from

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The system's **average energy** $\langle E \rangle$ is the expected value of E_ψ,

$$\langle E \rangle = \sum_\psi p_\psi E_\psi = \frac{1}{Z} \sum_\psi \exp\left(-\frac{E_\psi}{kT}\right) E_\psi \,.$$

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Rearranging, we get

$$\boxed{\langle E \rangle - TS = -kT \ln Z \equiv F} \, ,$$

which defines the **free energy** F.

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$$p_\psi = \frac{1}{Z} \exp\left(-\frac{E_\psi}{kT}\right) \, ,$$

we find

$$S = -k \sum_{\psi} p_\psi \left(-\frac{E_\psi}{kT} - \ln Z\right) = \frac{\langle E \rangle}{T} + k \ln Z \, .$$

Rearranging, we get

$$\boxed{\langle E \rangle - TS = -kT \ln Z \equiv F} \, ,$$

which defines the **free energy** F.

Fact. $S = -\dfrac{dF}{dT}$

4. The Main Calculation

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n_3 excitations at frequency 3ω, *etc.*,

has energy

$$E_\psi = \hbar\omega N \,,$$

where

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This string state corresponds to the partition

$$N = \overbrace{1 + 1 + \cdots + 1}^{n_1 \text{ summands}} + \overbrace{2 + 2 + \cdots + 2}^{n_2 \text{ summands}} + \overbrace{3 + 3 + \cdots + 3}^{n_3 \text{ summands}} + \cdots\,.$$

Therefore, counting the number of partitions of N is equivalent to counting the number of string states with energy $N\hbar\omega$.

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Strategy: Use statistical mechanics to calculate the entropy of the string, and thereby to count the number of partitions of N.

Everything begins with the partition function

$$Z = \sum_{\psi} \exp\left(-\frac{E_\psi}{kT}\right)$$

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Note. If we define $x = e^{-\frac{\hbar\omega}{kT}}$, then $Z = \prod_{\ell=1}^{\infty} \frac{1}{1 - x^\ell} = \sum_{N=0}^{\infty} p(N) x^N$ is exactly the generating function for partitions!

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We now take the limit $T \to \infty$, or equivalently, set $\dfrac{\hbar\omega}{kT} \ll 1$.

We'll see later that this is tantamount to assuming $N \gg 1$.

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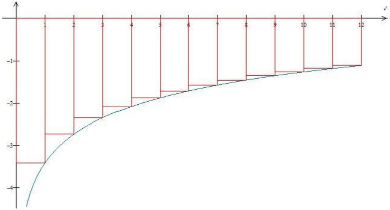
In this limit, the summand $\ln \left[1 - \exp\left(-\frac{\hbar\omega\ell}{kT}\right)\right]$ changes slowly with ℓ.



So we may approximate the sum by an integral,

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The entropy is

$$S = -\frac{dF}{dT} = -\frac{d}{dT}\left(-\frac{(kT)^2}{\hbar\omega} \cdot \frac{\pi^2}{6}\right) = \frac{k^2 T}{\hbar\omega} \cdot \frac{\pi^2}{3} \, .$$

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Written as a function of the energy, the entropy is

$$S(E) = k\pi\sqrt{\frac{2E}{3\hbar\omega}}\,.$$

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or

$$\boxed{\ln p(N) = 2\pi\sqrt{\frac{N}{6}}}\,, \qquad Q.E.D.!$$