Rank and nullity of Partition Regular Matrices

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September 6, 2009

1 Introduction

Terminology:

- Let $A \in \mathbb{Q}^{u\times v}$. We refer to A as being CC(m) if A satisfies the columns condition with a partition consistent of m classes.
- By Rado's theorem we have that A is CC(m) for some $m \in \mathbb{N}$ if and only if A is *partition regular*. We will use the abbreviation PR for partition regular and use it interchangeably with the statement "A is CC(m) for some $m \in \mathbb{N}$."
- Let $A \in \mathbb{Q}^{u\times v}$. An index k is a *null index* if for every vector $\mathbf{x} = [x_i] \in \ker A$, $x_k = 0$.

Observation 1.1.

1. *$A \in \mathrm{CC}(1)$ if and only if $A\mathbb{1} = \mathbf{0}$, where $\mathbb{1} = [1, \ldots, 1]^T$.*
2. *If A is CC(m), then rank(A) $\leq v - m$, since in this case A has at least m dependent columns.*
3. *If $\vec{x} \in \ker(A)$ such that no coordinate of $\vec{x}$ is allowed to be 0, and $D = \mathrm{diag}(x_1, \ldots, x_v)$, then the matrices $D^{-1}AD$ and AD are $CC(1)$ and therefore PR.*

Theorem 1.2. *If A is a $u \times v$ matrix such that A has a null index, then A is not PR.*

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2 Incidence matrices of oriented graphs

Adjacency matrices are not PR since they are nonnegative and nonzero matrices. Oriented vertex-edge incidence matrices can be PR and we look here at questions that arise naturally in this context.
Notation:
Let $\vec{G} = (V, E)$ be an oriented graph. Then $D_{\vec{G}}$ denotes its vertex-edge incidence matrix.

Observation 2.1. *While an arbitrary matrix A is $CC(1)$ if and only if $\vec{1} \in \ker(A)$, for any oriented graph $\vec{G}$, the vector $\mathbb{1} = [1, \ldots, 1]^T$ is in the left null space of $D_{\vec{G}}$.*

Observation 2.2. *Let $\vec{G}$ be an oriented graph and $\vec{D}_G$ its vertex-edge incidence matrix. If $\vec{G}$ has either a source or a sink then $\vec{D}_G$ is not PR.*

Observation 2.3. *For any $\vec{G}$, the matrix $D_{\vec{G}}$ has net column weight of 0. Note that the row weight is variable.*

Theorem 2.4. *Let G be a connected graph. The following are equivalent:*

1. *$K_e(G) \geq 2$, i.e., G has no bridge,*
2. *G is the union of its cycles,*
3. *G can be oriented so that the corresponding vertex-edge incidence matrix is PR.*

Proof. The following recursive algorithm produces the desired partition: Pick an unoriented cycle and orient it cyclically. Let the corresponding columns of the vertex-edge incidence matrix be the first cell of our partition I_1. Repeat this process until all edges are in some class I_k. □

Theorem 2.5. *For an oriented graph $\vec{G}$, the matrix $\vec{D}_G$ is PR if and only if $\vec{G}$ is strongly connected.*

Corollary 2.6. *For any $D_{\vec{G}}$, where $\vec{G}$ is strongly connected,*

$$\operatorname{rank} D_{\vec{G}} \leq |G| - 1.$$

Observation 2.7. *If $\vec{C_n}$ is an oriented cycle on n vertices then $D_{\vec{C_n}}$ is $CC(1)$ and $\operatorname{rank} D_{\vec{C_n}} = n - 1$.*

Theorem 2.8. *Let G be any graph which contains a Hamiltonian cycle. Then G can be oriented in such a way that $D_{\vec{G}}$ is $CC(2)$ and therefore PR.*

Theorem 2.9. *If $\vec{D^1}_G$ and $\vec{D^2}_G$ are two distinct orientated vertex-edge incidence matrices associated with a graph G, then there exists a signature matrix S such that $\vec{D^1}_G = \vec{D^2}_G \cdot S$.*

Observation 2.10. *Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. If $\mathbb{I} = \{I_1, \ldots, I_k\}$ is a partition of the the columns of $D_{\vec{G}}$ which satisfies the columns condition, then I_1 is an edge-disjoint union of cycles.*

Theorem 2.11. *Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. Any set $\{I_1, \ldots, I_t\}$ such that $I_i \in E_{\vec{G}}$, $I_j \cap I_l = \emptyset$ if $j \neq l$, and for all $1 \leq j \leq t, \sum_{i \in I_j} a_i \in \langle a_i : i \in \cup_{l=1}^{t} I_l \rangle$, can be extended to a partition of $E_{\vec{G}}$ that satisfies the columns condition.*

Corollary 2.12. *Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. The greedy algorithm produces a partition of the columns of $D_{\vec{G}}$ which satisfies the columns condition.*

3 Sign Patterns

A *sign pattern matrix* (or *sign pattern* for short) is a (rectangular) matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\mathrm{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. If $\mathbb{Y}$ is an $n \times n$ sign pattern, the *sign pattern class* (or *qualitative class*) of $\mathbb{Y}$, denoted $\mathcal{Q}(\mathbb{Y})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\mathrm{sgn}(A) = \mathbb{Y}$. It is traditional in the study of sign patterns to say that a sign pattern $\mathbb{Y}$ *requires* property P if every matrix in $\mathcal{Q}(\mathbb{Y})$ has property P and to say that $\mathbb{Y}$ *allows* property P if there exists a matrix in $\mathcal{Q}(\mathbb{Y})$ that has property P. Patterns that require partition regularity are too trivial to be of interest:

Theorem 3.1. *The only sign patterns that requires partition regularity are the all zero sign patterns.*

Proof. Assume $\mathbb{Y} = [\psi_{ij}]$ has a nonzero entry. Construct a matrix $A = [a_{ij}]$ as follows:

- For all i, j such that $\psi_{ij} = 0$, $a_{ij} = 0$.
- For all i, j such that $\psi_{ij} = +$, $a_{ij} = 1$.
- For all i, j such that $\psi_{ij} = -$, $a_{ij} = -\frac{1}{n}$.

There is no subset of columns that sum to zero, so A does not have the columns condition and so is not partition regular. □

Since a partition regular matrix must satisfy the columns condition, it is clear that in order for a sign pattern to allow partition regularity, any nonzero row must have both at least one $+$ entry and at least one $-$ entry. This is also sufficient for a sign pattern to allow partition regularity:

Theorem 3.2. *Let $\mathbb{Y}$ be an $m \times n$ sign pattern. The following are equivalent:*

1. *Each row of $\mathbb{Y}$ has at least one $+$ entry and at least one $-$ entry or every entry is* 0.
2. $\mathbb{Y}$ *allows* CC(1).
3. $\mathbb{Y}$ *allows partition regularity.*

Proof. It is clear that (2) $\implies$ (3) $\implies$ (1). Assume each row of $\mathbb{Y} = [\psi_{ij}]$ has at least one $+$ entry and at least one $-$ entry or every entry is 0. If row i is non entirely 0, let $m(i)$ denote the column number of the first $-$ entry in row i; otherwise, $m(i) = 0$. Construct a matrix $A = [a_{ij}]$ as follows:

- For all i, j such that $\psi_{ij} = 0$, $a_{ij} = 0$.
- For all i such that $m(i) > 0$:
  - If $\psi_{ij} = +$, $a_{ij} = 1$.
  - For $j > m(i)$, if $\psi_{ij} = -$ then $a_{ij} = -\frac{1}{n}$.
  - $a_{i,m(i)} = -\sum_{j\neq m(i)} a_{ij}$.

Clearly $A \in \mathcal{Q}(\mathbb{Y})$ and $A\mathbb{1} = \mathbf{0}$, so A has CC(1). □

The *minimum rank* of an $m \times n$ sign pattern $\mathbb{Y}$ is

$$\mathrm{mr}(\mathbb{Y}) = \min\{\mathrm{rank}(A) : \ A \in \mathcal{Q}(\mathbb{Y})\},$$

and the *maximum nullity* of $\mathcal{Y}$ is

$$\mathrm{M}(\mathbb{Y}) = \max\{\mathrm{null}(A) : \ A \in \mathcal{Q}(\mathbb{Y})\}.$$

Clearly $\mathrm{mr}(\mathcal{Y}) + \mathrm{M}(\mathcal{Y}) = n$.

It is not always the case that the nullity of a partition regulaar matrix can be realized by the number of cells in a columns condition partition. For example, for $A = \begin{bmatrix} 3 & -1 & -1 & -1 \end{bmatrix}$, $\mathrm{null}(A) = 3$ but A is $\mathrm{CC}(m)$ only for $m = 1$. But the following question remains open:

Question 3.3. *If $\mathbb{Y}$ allows partition regularity, must there exist a matrix $A \in \mathcal{Q}(\mathbb{Y})$ such that A is $\mathrm{CC}(\mathrm{M}(\mathbb{Y}))$?*

Theorem 3.4. *Let G be a connected graph, let $\vec{G}$ be an orientation of G, and let $\mathbb{Y} = \mathrm{sgn}(D_{\vec{G}})$. Then the following are equivalent:*

1. *$\mathbb{Y}$ allows PR*
2. *$D_{\vec{G}}$ is PR*
3. *$\vec{G}$ is strongly connected.*

Proof. [will write this later] □