Pythagoras at the Bat: An Introduction to Statistics and Mathematical Modeling

Steven J Miller
Williams College

sjm1@williams.edu
http://www.williams.edu/go/math/sjmiller

Wellesley College, September 21, 2009.

Acknowledgements

Sal Baxamusa, Phil Birnbaum, Chris Chiang, Ray Ciccolella, Steve Johnston, Michelle Manes, Russ Mann, students of Math 162 and Math 197 at Brown, 399 at Williams.

Dedicated to my great uncle Newt Bromberg (a lifetime Red Sox fan who promised me that I would live to see a World Series Championship in Boston).

Chris Long and the San Diego Padres.

Goals of the Talk

- Derive James’ Pythagorean Won-Loss formula from a reasonable model.
- Introduce some of the techniques of modeling.
- Discuss the mathematics behind the models and model testing.
- Show how advanced theory enters in simple problems.
- Further avenues for research for students.

Numerical Observation: Pythagorean Won-Loss Formula

Parameters

- $\mathrm{RS}_{\mathrm{obs}}$: average number of runs scored per game;
- $\mathrm{RA}_{\mathrm{obs}}$: average number of runs allowed per game;
- γ: some parameter, constant for a sport.

Numerical Observation: Pythagorean Won-Loss Formula

Parameters

- $\mathrm{RS}_{\mathrm{obs}}$: average number of runs scored per game;
- $\mathrm{RA}_{\mathrm{obs}}$: average number of runs allowed per game;
- γ: some parameter, constant for a sport.

James' Won-Loss Formula (NUMERICAL Observation)

$$\text{Won} - \text{Loss Percentage} = \frac{\mathrm{RS}_{\mathrm{obs}}{}^{\gamma}}{\mathrm{RS}_{\mathrm{obs}}{}^{\gamma} + \mathrm{RA}_{\mathrm{obs}}{}^{\gamma}}$$

γ originally taken as 2, numerical studies show best γ is about 1.82.

Applications of the Pythagorean Won-Loss Formula

- Extrapolation: use half-way through season to predict a team’s performance.
- Evaluation: see if consistently over-perform or under-perform.
- Advantage: Other statistics / formulas (run-differential per game); this is easy to use, depends only on two simple numbers for a team.

Probability Review

- Probability density:
  - ◇ $p(x) \geq 0$;
  - ◇ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ X random variable with density $p(x)$:

$$\mathrm{Prob}\,(X \in [a, b]) = \int_a^b p(x)\mathrm{d}x.$$

Probability Review

- Probability density:
  - ◇ $p(x) \geq 0$;
  - ◇ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ X random variable with density $p(x)$:

$$\mathrm{Prob}\left(X \in [a,b]\right) = \int_a^b p(x)\mathrm{d}x.$$

- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.

Probability Review

- Probability density:
  - ◇ $p(x) \geq 0$;
  - ◇ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ X random variable with density $p(x)$:

    $$\mathrm{Prob}\,(X \in [a,b]) = \int_a^b p(x)\mathrm{d}x.$$
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.

Probability Review

- Probability density:
  - ◇ $p(x) \geq 0$;
  - ◇ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ X random variable with density $p(x)$:

$$\text{Prob}\,(X \in [a,b]) = \int_a^b p(x)\mathrm{d}x.$$

- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: two random variables are independent if knowledge of one does not give knowledge of the other.

Modeling the Real World

Guidelines for Modeling:

- Model should capture key features of the system;
- Model should be mathematically tractable (solvable).

Modeling the Real World

Guidelines for Modeling:

- Model should capture key features of the system;
- Model should be mathematically tractable (solvable).

In general these are conflicting goals.

How should we try and model baseball games?

Modeling the Real World (cont)

Possible Model:

- Runs Scored and Runs Allowed independent random variables;
- $f_{\mathrm{RS}}(x)$, $g_{\mathrm{RA}}(y)$: probability density functions for runs scored (allowed).

Modeling the Real World (cont)

Possible Model:

- Runs Scored and Runs Allowed independent random variables;
- $f_{\mathrm{RS}}(x)$, $g_{\mathrm{RA}}(y)$: probability density functions for runs scored (allowed).

Reduced to calculating

$$\int_x \left[\int_{y \le x} f_{\mathrm{RS}}(x) g_{\mathrm{RA}}(y) \mathrm{d}y\right] \mathrm{d}x \quad \text{or} \quad \sum_i \left[\sum_{j<i} f_{\mathrm{RS}}(i) g_{\mathrm{RA}}(j)\right].$$

Problems with the Model

Reduced to calculating

$$\int_x \left[\int_{y \leq x} f_{\mathrm{RS}}(x) g_{\mathrm{RA}}(y) \mathrm{d}y \right] \mathrm{d}x \quad \text{or} \quad \sum_i \left[\sum_{j<i} f_{\mathrm{RS}}(i) g_{\mathrm{RA}}(j) \right].$$

Problems with the Model

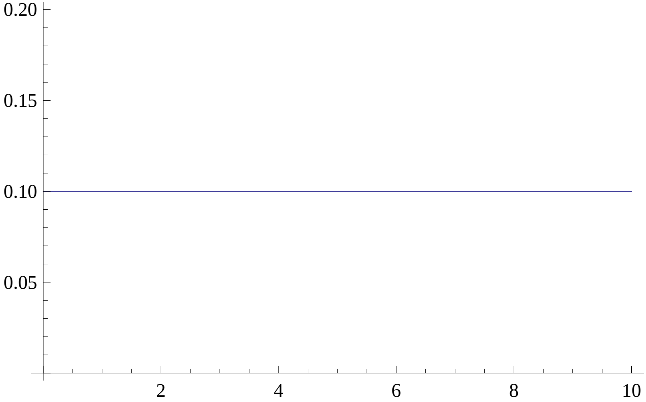
Reduced to calculating

$$\int_x \left[\int_{y \le x} f_{\mathrm{RS}}(x) g_{\mathrm{RA}}(y) \mathrm{d}y \right] \mathrm{d}x \quad \text{or} \quad \sum_i \left[\sum_{j<i} f_{\mathrm{RS}}(i) g_{\mathrm{RA}}(j) \right].$$

Problems with the model:

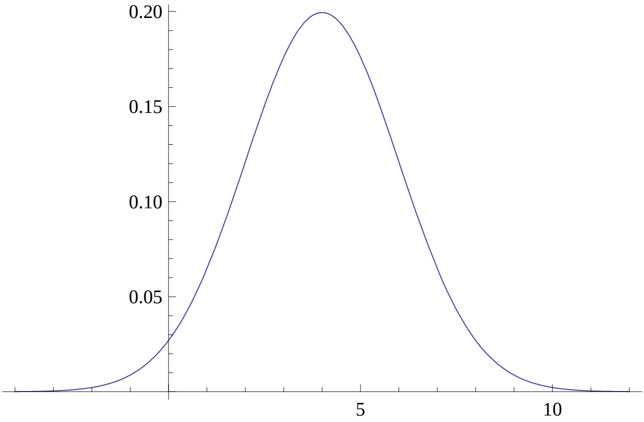
- Can the integral (or sum) be completed in closed form?
- Are the runs scored and allowed independent random variables?
- What are f_{RS} and g_{RA}?

Choices for f_{RS} and g_{RA}



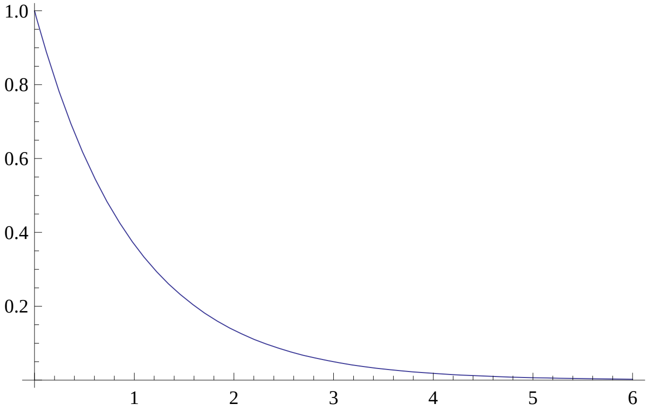
Uniform Distribution on $[0, 10]$.

Choices for f_{RS} and g_{RA}



Normal Distribution: mean 4, standard deviation 2.

Choices for f_{RS} and g_{RA}



Exponential Distribution: e^{-x}.

Three Parameter Weibull

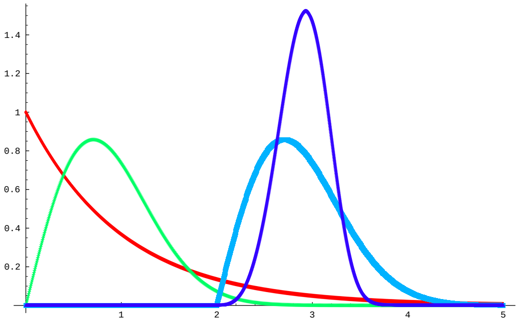
Weibull distribution:

$$f(x;\alpha,\beta,\gamma) = \begin{cases} \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} & \text{if } x \geq \beta \\ 0 & \text{otherwise.} \end{cases}$$

- α: scale (variance: meters versus centimeters);
- β: origin (mean: translation, zero point);
- γ: shape (behavior near β and at infinity).

Various values give different shapes, but can we find α, β, γ such that it fits observed data? Is the Weibull theoretically tractable?

Weibull Plots: Parameters (α, β, γ)



Red:(1, 0, 1) (exponential); Green:(1, 0, 2); Cyan:(1, 2, 2); Blue:(1, 2, 4)

Gamma Distribution

- For $s \in \mathbb{C}$ with the real part of s greater than 0, define the Γ-function:

$$\Gamma(s) \ = \ \int_0^\infty e^{-u} u^{s-1} \mathrm{d}u \ = \ \int_0^\infty e^{-u} u^s \frac{\mathrm{d}u}{u}.$$

- Generalizes factorial function: $\Gamma(n) = (n-1)!$ for $n \geq 1$ an integer.

Weibull Integrations

$$\mu_{\alpha,\beta,\gamma} = \int_{\beta}^{\infty} x \cdot \frac{\gamma}{\alpha} \left(\frac{x - \beta}{\alpha} \right)^{\gamma - 1} e^{-((x-\beta)/\alpha)^{\gamma}} \mathrm{d}x$$

Weibull Integrations

$$\begin{aligned}\mu_{\alpha,\beta,\gamma} &= \int_{\beta}^{\infty} x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} \mathrm{d}x \\ &= \int_{\beta}^{\infty} \alpha \frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} \mathrm{d}x \ + \ \beta.\end{aligned}$$

Weibull Integrations

$$\begin{aligned}\mu_{\alpha,\beta,\gamma} &= \int_\beta^\infty x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x \\ &= \int_\beta^\infty \alpha\frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x \ + \ \beta.\end{aligned}$$

Change variables: $u = \left(\frac{x-\beta}{\alpha}\right)^\gamma$, so $\mathrm{d}u = \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} \mathrm{d}x$

Weibull Integrations

$$\mu_{\alpha,\beta,\gamma} = \int_\beta^\infty x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x$$

$$= \int_\beta^\infty \alpha\frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x \;+\; \beta.$$

Change variables: $u = \left(\frac{x-\beta}{\alpha}\right)^\gamma$, so $\mathrm{d}u = \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1}\mathrm{d}x$ and

$$\mu_{\alpha,\beta,\gamma} = \int_0^\infty \alpha u^{\gamma^{-1}} \cdot e^{-u}\mathrm{d}u \;+\; \beta$$

Weibull Integrations

$$\mu_{\alpha,\beta,\gamma} = \int_{\beta}^{\infty} x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} \mathrm{d}x$$

$$= \int_{\beta}^{\infty} \alpha\frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} \mathrm{d}x \; + \; \beta.$$

Change variables: $u = \left(\frac{x-\beta}{\alpha}\right)^{\gamma}$, so $\mathrm{d}u = \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1}\mathrm{d}x$ and

$$\mu_{\alpha,\beta,\gamma} = \int_{0}^{\infty} \alpha u^{\gamma^{-1}} \cdot e^{-u}\mathrm{d}u \; + \; \beta$$

$$= \alpha \int_{0}^{\infty} e^{-u}\, u^{1+\gamma^{-1}} \frac{\mathrm{d}u}{u} \; + \; \beta$$

Weibull Integrations

$$\begin{aligned}\mu_{\alpha,\beta,\gamma} &= \int_\beta^\infty x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x \\ &= \int_\beta^\infty \alpha\frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} \mathrm{d}x \ + \ \beta.\end{aligned}$$

Change variables: $u = \left(\frac{x-\beta}{\alpha}\right)^\gamma$, so $\mathrm{d}u = \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1}\mathrm{d}x$ and

$$\begin{aligned}\mu_{\alpha,\beta,\gamma} &= \int_0^\infty \alpha u^{\gamma^{-1}} \cdot e^{-u} \mathrm{d}u \ + \ \beta \\ &= \alpha \int_0^\infty e^{-u}\, u^{1+\gamma^{-1}}\, \frac{\mathrm{d}u}{u} \ + \ \beta \\ &= \alpha\Gamma(1+\gamma^{-1}) \ + \ \beta.\end{aligned}$$

A similar calculation determines the variance.

Pythagorean Won-Loss Formula

Theorem (Pythagorean Won-Loss Formula)

Let the runs scored and allowed per game be two independent random variables drawn from Weibull distributions $(\alpha_{\rm RS}, \beta, \gamma)$ *and* $(\alpha_{\rm RA}, \beta, \gamma)$*;* $\alpha_{\rm RS}$ *and* $\alpha_{\rm RA}$ *are chosen so that the means are* RS *and* RA*. If* $\gamma > 0$ *then*

$$\text{Won-Loss Percentage}(\mathrm{RS}, \mathrm{RA}, \beta, \gamma) = \frac{(\mathrm{RS}-\beta)^\gamma}{(\mathrm{RS}-\beta)^\gamma + (\mathrm{RA}-\beta)^\gamma}$$

Pythagorean Won-Loss Formula

Theorem (Pythagorean Won-Loss Formula)

Let the runs scored and allowed per game be two independent random variables drawn from Weibull distributions $(\alpha_{\rm RS}, \beta, \gamma)$ *and* $(\alpha_{\rm RA}, \beta, \gamma)$*;* $\alpha_{\rm RS}$ *and* $\alpha_{\rm RA}$ *are chosen so that the means are* RS *and* RA*. If* $\gamma > 0$ *then*

$$\text{Won-Loss Percentage}(\text{RS}, \text{RA}, \beta, \gamma) = \frac{(\text{RS} - \beta)^\gamma}{(\text{RS} - \beta)^\gamma + (\text{RA} - \beta)^\gamma}$$

In baseball take $\beta = -1/2$ (from runs must be integers). $\text{RS} - \beta$ estimates average runs scored, $\text{RA} - \beta$ estimates average runs allowed.

Best Fit Weibulls to Data: Method of Least Squares

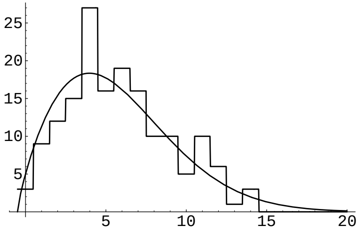
- $\mathrm{Bin}(k)$ is the k^{th} bin;
- $\mathrm{RS}_{\mathrm{obs}}(k)$ (resp. $\mathrm{RA}_{\mathrm{obs}}(k)$) the observed number of games with the number of runs scored (allowed) in $\mathrm{Bin}(k)$;
- $A(\alpha, \beta, \gamma, k)$ the area under the Weibull with parameters (α, β, γ) in $\mathrm{Bin}(k)$.

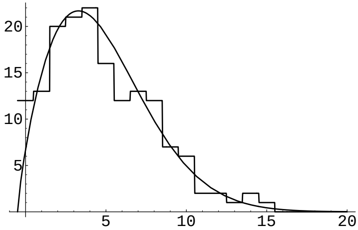
Find the values of $(\alpha_{\mathrm{RS}}, \alpha_{\mathrm{RA}}, \gamma)$ that minimize

$$\sum_{k=1}^{\#\mathrm{Bins}} \left(\mathrm{RS}_{\mathrm{obs}}(k) - \#\mathrm{Games} \cdot A(\alpha_{\mathrm{RS}}, -1/2, \gamma, k)\right)^2$$
$$+ \sum_{k=1}^{\#\mathrm{Bins}} \left(\mathrm{RA}_{\mathrm{obs}}(k) - \#\mathrm{Games} \cdot A(\alpha_{\mathrm{RA}}, -1/2, \gamma, k)\right)^2.$$

Best Fit Weibulls to Data (Method of Maximum Likelihood)

Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Boston Red Sox

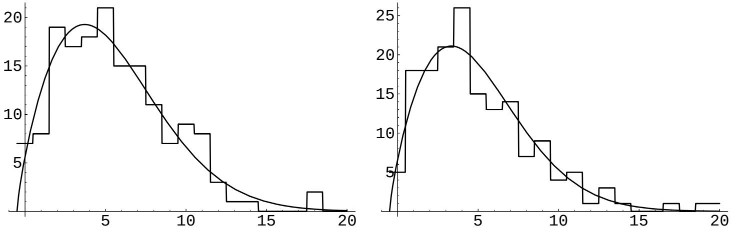




Using as bins $[-.5, .5] \cup [.5, 1.5] \cup \cdots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty)$.

Best Fit Weibulls to Data (Method of Maximum Likelihood)

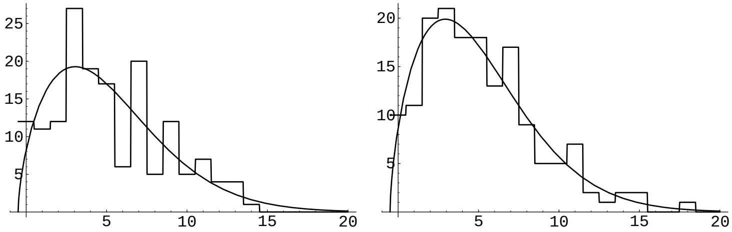
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the New York Yankees



Using as bins $[-.5, .5] \cup [.5, 1.5] \cup \cdots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty)$.

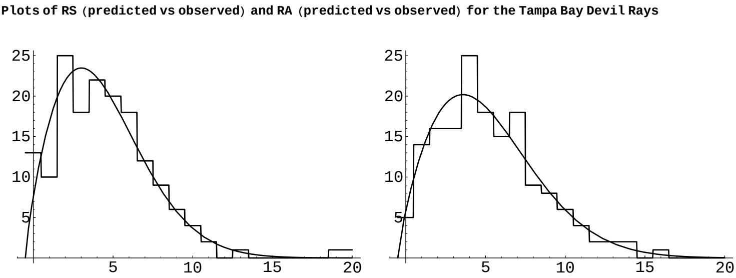
Best Fit Weibulls to Data (Method of Maximum Likelihood)

Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Baltimore Orioles



Using as bins $[-.5, .5] \cup [.5, 1.5] \cup \cdots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty)$.

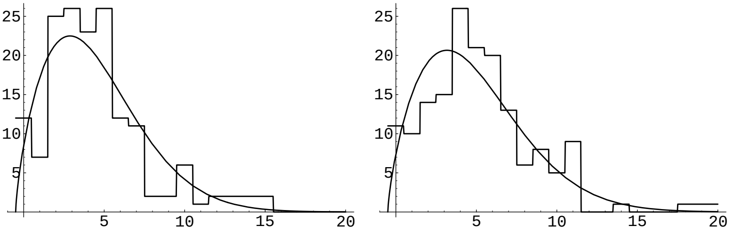
Best Fit Weibulls to Data (Method of Maximum Likelihood)



Using as bins $[-.5, .5] \ \cup \ [.5, 1.5] \ \cup \ \cdots \ \cup \ [7.5, 8.5]$ $\cup \ [8.5, 9.5] \ \cup [9.5, 11.5] \ \cup [11.5, \infty)$.

Best Fit Weibulls to Data (Method of Maximum Likelihood)

Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Toronto Blue Jays



Using as bins $[-.5, .5] \cup [.5, 1.5] \cup \cdots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty)$.

Bonferroni Adjustments

Fair coin: 1,000,000 flips, expect 500,000 heads.

Bonferroni Adjustments

Fair coin: 1,000,000 flips, expect 500,000 heads.
About 95% have $499,000 \le \#\text{Heads} \le 501,000$.

Bonferroni Adjustments

Fair coin: 1,000,000 flips, expect 500,000 heads.
About 95% have $499,000 \leq \#\text{Heads} \leq 501,000$.

Consider N independent experiments of flipping a fair coin 1,000,000 times. *What is the probability that at least one of set doesn't have* $499,000 \leq \#\text{Heads} \leq 501,000$*?*

N	Probability
5	22.62
14	51.23
50	92.31

See unlikely events happen as N increases!

Data Analysis: χ^2 Tests

Team	RS+RA χ2: 20 d.f.	Indep χ2: 109 d.f
Boston Red Sox	15.63	83.19
New York Yankees	12.60	129.13
Baltimore Orioles	29.11	116.88
Tampa Bay Devil Rays	13.67	111.08
Toronto Blue Jays	41.18	100.11
Minnesota Twins	17.46	97.93
Chicago White Sox	22.51	153.07
Cleveland Indians	17.88	107.14
Detroit Tigers	12.50	131.27
Kansas City Royals	28.18	111.45
Los Angeles Angels	23.19	125.13
Oakland Athletics	30.22	133.72
Texas Rangers	16.57	111.96
Seattle Mariners	21.57	141.00

20 d.f.: 31.41 (at the 95% level) and 37.57 (at the 99% level).
109 d.f.: 134.4 (at the 95% level) and 146.3 (at the 99% level).
Bonferroni Adjustment:
20 d.f.: 41.14 (at the 95% level) and 46.38 (at the 99% level).
109 d.f.: 152.9 (at the 95% level) and 162.2 (at the 99% level).

Data Analysis: Structural Zeros

- For independence of runs scored and allowed, use bins $[0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup [8,9) \cup [9,10) \cup [10,11) \cup [11,\infty)$.

- Have an $r \times c$ contingency table with structural zeros (runs scored and allowed per game are never equal).

- (Essentially) $O_{r,r} = 0$ for all r, use an iterative fitting procedure to obtain maximum likelihood estimators for $E_{r,c}$ (expected frequency of cell (r,c) assuming that, given runs scored and allowed are distinct, the runs scored and allowed are independent).

Testing the Model: Data from Method of Maximum Likelihood

Team	Obs Wins	Pred Wins	ObsPerc	PredPerc	GamesDiff	γ
Boston Red Sox	98	93.0	0.605	0.574	5.03	1.82
New York Yankees	101	87.5	0.623	0.540	13.49	1.78
Baltimore Orioles	78	83.1	0.481	0.513	−5.08	1.66
Tampa Bay Devil Rays	70	69.6	0.435	0.432	0.38	1.83
Toronto Blue Jays	67	74.6	0.416	0.464	−7.65	1.97
Minnesota Twins	92	84.7	0.568	0.523	7.31	1.79
Chicago White Sox	83	85.3	0.512	0.527	−2.33	1.73
Cleveland Indians	80	80.0	0.494	0.494	0.	1.79
Detroit Tigers	72	80.0	0.444	0.494	−8.02	1.78
Kansas City Royals	58	68.7	0.358	0.424	−10.65	1.76
Los Angeles Angels	92	87.5	0.568	0.540	4.53	1.71
Oakland Athletics	91	84.0	0.562	0.519	6.99	1.76
Texas Rangers	89	87.3	0.549	0.539	1.71	1.90
Seattle Mariners	63	70.7	0.389	0.436	−7.66	1.78

γ: mean $=$ 1.74, standard deviation $=$ .06, median $=$ 1.76; close to numerically observed value of 1.82.

Conclusions

- Find parameters such that Weibulls are good fits;
- Runs scored and allowed per game are statistically independent;
- Pythagorean Won-Loss Formula is a consequence of our model;
- Best γ (both close to observed best 1.82):
  ⋄ Method of Least Squares: 1.79;
  ⋄ Method of Maximum Likelihood: 1.74.

Future Work

- Micro-analysis: runs scored and allowed are not entirely independent (big lead, close game), run production smaller for inter-league games in NL parks, et cetera.

- Other sports: Does the same model work? How does γ depend on the sport?

- Closed forms: Are there other probability distributions that give integrals which can be determined in closed form?

- Valuing Runs: Pythagorean formula used to value players (10 runs equals 1 win); better model leads to better team.

References

- Baxamusa, Sal:
  ◇ Weibull worksheet: http://www.beyondtheboxscore.com/story/2006/4/30/114737/251

  ◇ Run distribution plots for various teams:
  http://www.beyondtheboxscore.com/story/2006/2/23/164417/484

- Miller, Steven J.:
  ◇ *A Derivation of James' Pythagorean projection*, By The Numbers – The Newsletter of the SABR Statistical Analysis Committee, vol. 16 (February 2006), no. 1, 17–22.
  http://www.philbirnbaum.com/btn2006-02.pdf

  ◇ *A derivation of the Pythagorean Won-Loss Formula in baseball*, Chance Magazine **20** (2007), no. 1, 40–48. http://www.williams.edu/go/math/sjmiller/public_html/math/talks/talk.html

Appendix I: Proof of the Pythagorean Won-Loss Formula

Let X and Y be independent random variables with Weibull distributions $(\alpha_{\rm RS}, \beta, \gamma)$ and $(\alpha_{\rm RA}, \beta, \gamma)$ respectively. To have means of $\rm RS - \beta$ and $\rm RA - \beta$ our calculations for the means imply

$$\alpha_{\rm RS} \ = \ \frac{\rm RS - \beta}{\Gamma(1+\gamma^{-1})}, \qquad \alpha_{\rm RA} \ = \ \frac{\rm RA - \beta}{\Gamma(1+\gamma^{-1})}.$$

We need only calculate the probability that X exceeds Y. We use the integral of a probability density is 1.

Appendix I: Proof of the Pythagorean Won-Loss Formula (cont)

$$\begin{aligned}
\text{Prob}(X > Y) &= \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} f(x;\alpha_{\mathrm{RS}},\beta,\gamma)f(y;\alpha_{\mathrm{RA}},\beta,\gamma)\mathrm{d}y\ \mathrm{d}x\\
&= \int_{\beta}^{\infty}\int_{\beta}^{x}\frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x-\beta}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x-\beta}{\alpha_{\mathrm{RS}}}\right)^{\gamma}}\frac{\gamma}{\alpha_{\mathrm{RA}}}\left(\frac{y-\beta}{\alpha_{\mathrm{RA}}}\right)^{\gamma-1} e^{-\left(\frac{y-\beta}{\alpha_{\mathrm{RA}}}\right)^{\gamma}}\mathrm{d}y\,\mathrm{d}x\\
&= \int_{x=0}^{\infty}\frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha_{\mathrm{RS}}}\right)^{\gamma}}\left[\int_{y=0}^{x}\frac{\gamma}{\alpha_{\mathrm{RA}}}\left(\frac{y}{\alpha_{\mathrm{RA}}}\right)^{\gamma-1} e^{-\left(\frac{y}{\alpha_{\mathrm{RA}}}\right)^{\gamma}}\mathrm{d}y\right]\mathrm{d}x\\
&= \int_{x=0}^{\infty}\frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{\mathrm{RS}})^{\gamma}}\left[1-e^{-(x/\alpha_{\mathrm{RA}})^{\gamma}}\right]\mathrm{d}x\\
&= 1-\int_{x=0}^{\infty}\frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha)^{\gamma}}\mathrm{d}x,
\end{aligned}$$

where we have set

$$\frac{1}{\alpha^{\gamma}} = \frac{1}{\alpha_{\mathrm{RS}}^{\gamma}}+\frac{1}{\alpha_{\mathrm{RA}}^{\gamma}} = \frac{\alpha_{\mathrm{RS}}^{\gamma}+\alpha_{\mathrm{RA}}^{\gamma}}{\alpha_{\mathrm{RS}}^{\gamma}\alpha_{\mathrm{RA}}^{\gamma}}.$$

Appendix I: Proof of the Pythagorean Won-Loss Formula (cont)

$$\begin{aligned}
\text{Prob}(X > Y) &= 1 - \frac{\alpha^\gamma}{\alpha_{\mathrm{RS}}^\gamma} \int_0^\infty \frac{\gamma}{\alpha} \left(\frac{x}{\alpha}\right)^{\gamma - 1} e^{(x/\alpha)^\gamma} dx \\
&= 1 - \frac{\alpha^\gamma}{\alpha_{\mathrm{RS}}^\gamma} \\
&= 1 - \frac{1}{\alpha_{\mathrm{RS}}^\gamma} \frac{\alpha_{\mathrm{RS}}^\gamma \alpha_{\mathrm{RA}}^\gamma}{\alpha_{\mathrm{RS}}^\gamma + \alpha_{\mathrm{RA}}^\gamma} \\
&= \frac{\alpha_{\mathrm{RS}}^\gamma}{\alpha_{\mathrm{RS}}^\gamma + \alpha_{\mathrm{RA}}^\gamma}.
\end{aligned}$$

We substitute the relations for α_{RS} and α_{RA} and find that

$$\text{Prob}(X > Y) = \frac{(\mathrm{RS} - \beta)^\gamma}{(\mathrm{RS} - \beta)^\gamma + (\mathrm{RA} - \beta)^\gamma}.$$

Note $\mathrm{RS} - \beta$ estimates $\mathrm{RS}_{\mathrm{obs}}$, $\mathrm{RA} - \beta$ estimates $\mathrm{RA}_{\mathrm{obs}}$.

Appendix II: Best Fit Weibulls and Structural Zeros

The fits *look* good, but are they? Do χ^2-tests:

- Let $\mathrm{Bin}(k)$ denote the k^{th} bin.
- $O_{r,c}$: the observed number of games where the team's runs scored is in $\mathrm{Bin}(r)$ and the runs allowed are in $\mathrm{Bin}(c)$.
- $E_{r,c} = \frac{\sum_{c'} O_{r,c'} \cdot \sum_{r'} O_{r',c}}{\#\mathrm{Games}}$ is the expected frequency of cell (r, c).
- Then

$$\sum_{r=1}^{\#\mathrm{Rows}} \sum_{c=1}^{\#\mathrm{Columns}} \frac{(O_{r,c} - E_{r,c})^2}{E_{r,c}}$$

is a χ^2 distribution with $(\#\mathrm{Rows} - 1)(\#\mathrm{Columns} - 1)$ degrees of freedom.

Appendix II: Best Fit Weibulls and Structural Zeros (cont)

For independence of runs scored and allowed, use bins

$$[0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup [8,9) \cup [9,10) \cup [10,11) \cup [11,\infty).$$

Have an $r \times c$ contingency table (with $r = c = 12$); however, there are *structural zeros* (runs scored and allowed per game can never be equal).

(Essentially) $O_{r,r} = 0$ for all r. We use the iterative fitting procedure to obtain maximum likelihood estimators for the $E_{r,c}$, the expected frequency of cell (r,c) under the assumption that, given that the runs scored and allowed are distinct, the runs scored and allowed are independent.

For $1 \leq r, c \leq 12$, let $E_{r,c}^{(0)} = 1$ if $r \neq c$ and 0 if $r = c$. Set

$$X_{r,+} = \sum_{c=1}^{12} O_{r,c}, \quad X_{+,c} = \sum_{r=1}^{12} O_{r,c}.$$

Then

$$E_{r,c}^{(\ell)} = \begin{cases} E_{r,c}^{(\ell-1)} X_{r,+} \ / \ \sum_{c=1}^{12} E_{r,c}^{(\ell-1)} & \text{if } \ell \text{ is odd} \\ E_{r,c}^{(\ell-1)} X_{+,c} \ / \ \sum_{r=1}^{12} E_{r,c}^{(\ell-1)} & \text{if } \ell \text{ is even,} \end{cases}$$

and

$$E_{r,c} = \lim_{\ell \to \infty} E_{r,c}^{(\ell)};$$

the iterations converge very quickly. (If we had a complete two-dimensional contingency table, then the iteration reduces to the standard values, namely $E_{r,c} = \sum_{c'} O_{r,c'} \cdot \sum_{r'} O_{r',c} \ / \ \#\text{Games}$.). Note

$$\sum_{r=1}^{12} \sum_{\substack{c=1 \\ c \neq r}}^{12} \frac{(O_{r,c} - E_{r,c})^2}{E_{r,c}}$$

Appendix III: Central Limit Theorem

Convolution of f and g:

$$h(y) \;=\; (f * g)(y) \;=\; \int_{\mathbb{R}} f(x)g(y-x)\mathrm{d}x \;=\; \int_{\mathbb{R}} f(x-y)g(x)\mathrm{d}x.$$

X_1 and X_2 independent random variables with probability density p.

$$\mathrm{Prob}(X_i \in [x, x+\Delta x]) \;=\; \int_x^{x+\Delta x} p(t)\mathrm{d}t \;\approx\; p(x)\Delta x.$$

$$\mathrm{Prob}(X_1 + X_2) \in [x, x+\Delta x] \;=\; \int_{x_1=-\infty}^{\infty} \int_{x_2=x-x_1}^{x+\Delta x - x_1} p(x_1)p(x_2)\mathrm{d}x_2\mathrm{d}x_1.$$

As $\Delta x \to 0$ we obtain the convolution of p with itself:

$$\mathrm{Prob}(X_1 + X_2 \in [a, b]) \;=\; \int_a^b (p * p)(z)\mathrm{d}z.$$

Exercise to show non-negative and integrates to 1.

Appendix III: Statement of Central Limit Theorem

- For simplicity, assume p has mean zero, variance one, finite third moment and is of sufficiently rapid decay so that all convolution integrals that arise converge: p an infinitely differentiable function satisfying

$$\int_{-\infty}^{\infty} xp(x)\mathrm{d}x = 0, \quad \int_{-\infty}^{\infty} x^2p(x)\mathrm{d}x = 1, \quad \int_{-\infty}^{\infty} |x|^3p(x)\mathrm{d}x < \infty.$$

- Assume $X_1, X_2, \dots$ are independent identically distributed random variables drawn from p.
- Define $S_N = \sum_{i=1}^{N} X_i$.
- Standard Gaussian (mean zero, variance one) is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Central Limit Theorem Let X_i, S_N be as above and assume the third moment of each X_i is finite. Then $S_N/\sqrt{N}$ converges in probability to the standard Gaussian:

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{S_N}{\sqrt{N}} \in [a,b]\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\mathrm{d}x.$$

Appendix III: Proof of the Central Limit Theorem

- The Fourier transform of p is
$$\widehat{p}(y) = \int_{-\infty}^{\infty} p(x)e^{-2\pi ixy}\mathrm{d}x.$$

- Derivative of $\widehat{g}$ is the Fourier transform of $2\pi ixg(x)$; differentiation (hard) is converted to multiplication (easy).
$$\widehat{g}'(y) = \int_{-\infty}^{\infty} 2\pi ix \cdot g(x)e^{-2\pi ixy}\mathrm{d}x.$$
If g is a probability density, $\widehat{g}'(0) = 2\pi i\mathbb{E}[x]$ and $\widehat{g}''(0) = -4\pi^2\mathbb{E}[x^2]$.

- Natural to use the Fourier transform to analyze probability distributions. The mean and variance are simple multiples of the derivatives of $\widehat{p}$ at zero: $\widehat{p}'(0) = 0$, $\widehat{p}''(0) = -4\pi^2$.

- We Taylor expand $\widehat{p}$ (need technical conditions on p):
$$\widehat{p}(y) = 1 + \frac{p''(0)}{2}y^2 + \cdots = 1 - 2\pi^2y^2 + O(y^3).$$
Near the origin, the above shows $\widehat{p}$ looks like a concave down parabola.

Appendix III: Proof of the Central Limit Theorem (cont)

- $\mathrm{Prob}(X_1 + \cdots + X_N \in [a, b]) \;=\; \int_a^b (p * \cdots * p)(z)\mathrm{d}z.$
- The Fourier transform converts convolution to multiplication. If $\mathrm{FT}[f](y)$ denotes the Fourier transform of f evaluated at y:
$$\mathrm{FT}[p * \cdots * p](y) \;=\; \widehat{p}(y) \cdots \widehat{p}(y).$$
- Do not want the distribution of $X_1 + \cdots + X_N = x$, but rather $S_N \;=\; \frac{X_1+\cdots+X_N}{\sqrt{N}} \;=\; x.$
- If $B(x) = A(cx)$ for some fixed $c \neq 0$, then $\widehat{B}(y) = \frac{1}{c}\widehat{A}\left(\frac{y}{c}\right)$.
- $\mathrm{Prob}\left(\frac{X_1+\cdots+X_N}{\sqrt{N}} = x\right) \;=\; (\sqrt{N}p * \cdots * \sqrt{N}p)(x\sqrt{N}).$
- $\mathrm{FT}\left[(\sqrt{N}p * \cdots * \sqrt{N}p)(x\sqrt{N})\right](y) \;=\; \left[\widehat{p}\left(\frac{y}{\sqrt{N}}\right)\right]^N.$

Appendix III: Proof of the Central Limit Theorem (cont)

- Can find the Fourier transform of the distribution of S_N:

$$\left[\widehat{p}\left(\frac{y}{\sqrt{N}}\right)\right]^N.$$

- Take the limit as $N \to \infty$ for **fixed** y.
- Know $\widehat{p}(y) = 1 - 2\pi^2 y^2 + O(y^3)$. Thus study

$$\left[1 - \frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N.$$

- For any **fixed** y,

$$\lim_{N\to\infty}\left[1 - \frac{2\pi^2 y^2}{N} + O\left(\frac{y^3}{N^{3/2}}\right)\right]^N = e^{-2\pi y^2}.$$

- Fourier transform of $e^{-2\pi y^2}$ at x is $\frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}$.

Appendix III: Proof of the Central Limit Theorem (cont)

We have shown:

- the Fourier transform of the distribution of S_N converges to $e^{-2\pi y^2}$;
- the Fourier transform of $e^{-2\pi y^2}$ is $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Therefore the distribution of S_N equalling x converges to $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.
We need complex analysis to justify this conclusion. Must be careful: Consider

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

All the Taylor coefficients about $x = 0$ are zero, but the function is not identically zero in a neighborhood of $x = 0$.

Appendix IV: Best Fit Weibulls from Method of Maximum Likelihood

The likelihood function depends on: $\alpha_{\rm RS}, \alpha_{\rm RA}, \beta = -.5, \gamma$.
Let $A(\alpha, -.5, \gamma, k)$ denote the area in $\mathrm{Bin}(k)$ of the Weibull with parameters $\alpha, -.5, \gamma$. The sample likelihood function $L(\alpha_{\rm RS}, \alpha_{\rm RA}, -.5, \gamma)$ is

$$\binom{\#\mathrm{Games}}{\mathrm{RS_{obs}}(1), \dots, \mathrm{RS_{obs}}(\#\mathrm{Bins})} \prod_{k=1}^{\#\mathrm{Bins}} A(\alpha_{\rm RS}, -.5, \gamma, k)^{\mathrm{RS_{obs}}(k)}$$

$$\cdot \binom{\#\mathrm{Games}}{\mathrm{RA_{obs}}(1), \dots, \mathrm{RA_{obs}}(\#\mathrm{Bins})} \prod_{k=1}^{\#\mathrm{Bins}} A(\alpha_{\rm RA}, -.5, \gamma, k)^{\mathrm{RA_{obs}}(k)}.$$

For each team we find the values of the parameters $\alpha_{\rm RS}$, $\alpha_{\rm RA}$ and γ that maximize the likelihood. Computationally, it is equivalent to maximize the logarithm of the likelihood, and we may ignore the multinomial coefficients are they are independent of the parameters.