

Problem 7 in the May 2006 Calendar says:

Suppose the numbers 1, 2, 4, 8, 16, 32, 64, 128, and 256 are placed into the squares in a  $3 \times 3$  grid in such a way that each of the 9 numbers appears exactly once and the product of the numbers appearing in any row or column is the same. What is the value of the product in each row and column?

The solution given with the calendar explains that the common product must be 4096 because each number is an integer power of 2 and the sum of the exponents in each row or column must be 12, but it does not explain how to place the nine numbers in the grid to satisfy this condition.

Here is a  $3 \times 3$  *magic square*.

8	1	6
3	5	7
4	9	2

It is called a magic square because it consists of consecutive positive integers starting with 1 and the sum (which is called the *magic constant*) of the numbers along any row, column, or main diagonal is always the same. For this magic square, the magic constant is 15. If 1 is subtracted from every number in the magic square, the sums will all be 12, which is just what is needed; and the desired grid looks like this:

$2^7$	$2^0$	$2^5$		$2^7$	$2^0$	$2^5$
$2^2$	$2^4$	$2^6$	$\Rightarrow$	128	1	32
$2^3$	$2^8$	$2^1$		4	16	64
				8	256	2

The product of the numbers appearing in any row or column is 4096, as required, and the product along either main diagonal is also 4096. This is called a *multiplication magic square* with magic constant 4096.

If squares differing by only rotations and reflections are considered identical, there is only one  $3 \times 3$  magic square, and hence there is only one  $3 \times 3$  multiplication magic square with magic constant 4096. If only the sums along rows and columns are required to be the same, the resulting grid is called a *semimagic square*. There are nine distinct  $3 \times 3$  semimagic squares, each with a different number at its center. From these, nine distinct grids can be created to satisfy the conditions specified in the problem. Here, for example, are two of the semimagic squares along with the solutions they generate for the calendar problem:

8	1	6	$\Rightarrow$	$2^7$	$2^0$	$2^5$	$\Rightarrow$	128	1	32
4	9	2		$2^3$	$2^8$	$2^1$		8	256	2
3	5	7		$2^2$	$2^4$	$2^6$		4	16	64

2	4	9	$\Rightarrow$	$2^1$	$2^3$	$2^8$	$\Rightarrow$	2	8	256
6	8	1		$2^5$	$2^7$	$2^0$		32	128	1
7	3	5		$2^6$	$2^2$	$2^4$		64	4	16

More information, including techniques for generating magic squares and a list of references, can be found at

<http://mathworld.wolfram.com/MagicSquare.html> ,

and a fascinating discussion of the properties of multiplication magic squares can be found at

<http://cboyer.club.fr/multimagie/English/Multiplicative.htm> .

**George M. Caplan**

*gcaplan@wellesley.edu*

Wellesley College

Wellesley, MA 02481-8203