Ye olde inertia demonstration

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A simple quantitative analysis of the classic inertia ball demonstration explains why the lower string may break for “jerks” weaker than those that normally break the upper string, and why both strings may break—first the lower, then the upper. © 2004 American Association of Physics Teachers. [DOI: 10.1119/1.1677337]

I. INTRODUCTION

A common introductory physics demonstration consists of a weight hung on a string with an identical string tied to the bottom of the weight (see Fig. 1).1–4 A slow pull on the lower string breaks the upper string, but a fast pull breaks the lower string, leaving the upper intact. The qualitative explanation is that the inertia of the mass prevents the upper string from breaking when the lower is pulled quickly. To our knowledge, only a few papers have analyzed this demonstration quantitatively.5–8 We discuss two surprising behaviors that we call sequential breaking and anomalous breaking.

In sequential breaking, both strings break—first the lower, then the upper. Sequential breaking occurs if, at the time that the lower string breaks, the mass has sufficient momentum to break the upper string. Sequential breaking was analyzed in Refs. 5 and 6. The former correctly analyzed sequential breaking for a special case. The latter generalized the analysis, but contains an error, which our analysis corrects.

In anomalous breaking, the lower string breaks for a pull slower than the pull required to break the upper string. Anomalous breaking, which was analyzed in Ref. 8, arises from the elastic behavior of the upper string, and it may be difficult to observe in practice. Our investigation offers an improved graphical method for understanding anomalous breaking.

II. THE MODEL

We choose a very simple model for our analysis.

1. The force applied to the lower string starts at zero and increases linearly with time. This assumption means that the force applied to the lower string has the form

\[ F_{\text{appl}}(t) = \alpha t \quad (t>0). \]

2. The strings are massless. This assumption means that \( F_{\text{appl}} \) can be considered to be applied directly to the mass \( m \), and the spring constant of the lower string need not be considered.9

3. The strings obey Hooke’s law up to failure at tension \( T_0 \). Thus, the tension \( T_{\text{up}} \) in the upper string is

\[ T_{\text{up}} = k \Delta x \quad (T_{\text{up}} < T_0), \]

where \( k \) and \( \Delta x \) are, respectively, the spring constant and the extension of the upper string.

Given this model, the equation of motion of the mass is

\[ \ddot{x} + (k/m)x = (\alpha/m)t, \]

where \( x \) is the downward displacement from the equilibrium position of the mass hanging on the upper string.

\[ x = \Delta x - mg/k. \]

For the standard initial conditions from rest at \( t=0 \), the solution is

\[ x = (\alpha/k)[t - \sqrt{m/k} \sin(\sqrt{k/m}t)] \quad (t>0). \]

The tension \( T_{\text{low}} \) in the lower string is simply the applied force,

\[ T_{\text{low}} = F_{\text{appl}}(t) = \alpha t. \]

The tension \( T_{\text{up}} \) in the upper string is

\[ T_{\text{up}} = mg + kx = mg + \alpha[t - \sqrt{m/k} \sin(\sqrt{k/m}t)]. \]

When \( x = 0 \), the tension in the upper string is \( mg \), the weight of the hanging mass. We call this tension the static loading.

III. ZONE BOUNDARIES

Given our model, we want to distinguish quantitatively between the upper string breaking first and the lower string breaking first. The boundaries between these two outcomes are zone boundaries. At the zone boundaries, the tensions in the two strings reach \( T_0 \) simultaneously. So, we set

\[ T_{\text{low}} = T_{\text{up}} = T_0. \]

We eliminate the two string tensions and the time \( t \) from Eqs. (6) to (8) and obtain

\[ \sin(1/J) / (1/J) = \gamma, \]

where \( \gamma \) is the normalized static loading.

\[ \gamma = mg/T_0, \]

and \( J \) is the normalized “jerk.”

\[ J = a/\left(T_0\sqrt{k/m}\right). \]
The resulting zone boundaries are shown as solid curves in Fig. 2, which plots $g$ against $J$. For a particular value of $g$, the value(s) of $J$ satisfying Eq. (9) can be found by drawing a horizontal line on Fig. 2 at the appropriate height. For sufficiently small values of $g$, Eq. (9) has multiple solutions that give rise to anomalous zones. For values of $g$ and $J$ within these zones, anomalous breaking will occur. There is one such zone (the principal anomalous zone) for $0.071 < g < 0.128$, two for $0.049 < g < 0.071$, etc. These values of $g$ appear in Ref. 8, but Fig. 2 and Eq. (9) make the calculation and interpretation of these values much clearer.

Figure 2 is similar to Fig. 1 in Ref. 6, but the anomalous zones were overlooked there.

As an example of anomalous breaking, consider $g = 0.08$. Because $J = 0.34$ satisfies Eq. (9) when $g = 0.08$, the lower string will break for $J > 0.34$. This result is expected for a fast pull on the lower string. The conventional qualitative analysis suggests that the upper string will break for all $J < 0.34$, but this result is not exactly what our model predicts. For $g = 0.08$, $J = 0.15$ and $J = 0.12$ also satisfy Eq. (9), that is, the lower string breaks for $0.12 < J < 0.15$ as well as for $J > 0.34$. Figure 6 shows a plot of the tensions for $g = 0.08$ and $J = 0.13$; in Fig. 2 this point is labeled “Fig. 6.”

IV. SEQUENTIAL BREAKING

When the lower string breaks, the mass may have sufficient momentum to cause the upper string to break later. From Eq. (6), the lower string breaks at the time $t_1 = T_0/\alpha$. If we use Eq. (5) and its derivative, we can evaluate the mass’s displacement and velocity at $t_1$ and then calculate the total energy $E$. The maximum subsequent displacement, $x_{\text{max}}$, of the mass can then be found from

$$E = \frac{1}{2} k x_{\text{max}}^2,$$

and it can be compared with the displacement $x = (T_0 - mg)/k$ at which the upper string breaks [from Eq. (7) with $T_{up} = T_0$]. In terms of the normalized variables of Eqs. (10) and (11), the boundary for subsequent breaking of the upper string is given by

$$\left[1 - (\sin(1/J)/(1/J))\right]^2 + \left[(1 - \cos(1/J))/(1/J)\right]^2 = (1 - \gamma)^2.$$

This boundary for sequential breaking is shown as the dashed curve in Fig. 2. An equation similar to Eq. (13) appears in Ref. 6, but contains an error; consequently, the boundary for sequential breaking is shown incorrectly in Fig. 1 of Ref. 6.
As an example of sequential breaking, consider $\gamma = 0.4$. Then $J = 0.47$ satisfies Eq. (9), and $J = 0.80$ satisfies Eq. (13). This solution means that the lower string will break for all $J > 0.47$; and for $0.47 < J < 0.80$, the upper string will break after the lower string breaks. Figure 4 shows a plot of the tensions for $\gamma = 0.4$ and $J = 0.6$; in Fig. 2 this point is labeled “Fig. 4.”

V. PLOTS OF THE TENSION

Figures 3–6 are time plots of the tensions in the two strings. The parameter values were chosen to represent the various zones in Fig. 2. The heavier curve is the upper string’s tension—before the lower string breaks, this tension is $J$; after the lower string breaks, it is the string’s tension. The lighter curve is the lower string’s tension from Eq. (5). As an example of sequential breaking, consider $\gamma = 0.4$. Then $J = 0.47$ satisfies Eq. (9), and $J = 0.80$ satisfies Eq. (13). This solution means that the lower string will break for all $J > 0.47$; and for $0.47 < J < 0.80$, the upper string will break after the lower string breaks. Figure 4 shows a plot of the tensions for $\gamma = 0.4$ and $J = 0.6$; in Fig. 2 this point is labeled “Fig. 4.”

VI. CONCLUSIONS

Our simple model predicts anomalous breaking (shown in Fig. 6) in which the lower string breaks “on the bounce” for some combinations of light static loading ($\gamma < 0.128$) and jerks weaker than those that normally break the upper string. As discussed in Ref. 8, this behavior may be difficult to observe in the laboratory—even if the “jerky” force is better controlled than in the typical lecture demonstration.

For intermediate values of jerk, the model predicts sequential breaking (shown in Fig. 4) in which the upper string breaks after the lower string has broken. Sequential breaking may be more easily observed in the laboratory. Indeed, if an instructor is not forewarned, the double breaking might be an unwanted surprise in a lecture demonstration!

One can imagine building some sort of mechanical device to provide well-controlled, reproducible “jerky” forces (or displacements\(^10\)) to demonstrate these two surprising and subtle behaviors.

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\(^10\)The *Video Encyclopedia of Physics Demonstrations* (The Education Group & Associates, Los Angeles, CA, 1992), Disc 2, Chap. 5, Demo 02-13, “Inertia Ball.”

This demonstration is described at (faraday.physics.uiowa.edu/mech/1F20.10.htm) and (www.physics.brown.edu/Studies/Demo/solids/demos/12010.html).

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**Fig. 5.** String tensions for $\gamma = 0.4$ and $J = 0.9$ in the normal high-jerk zone.

**Fig. 6.** String tensions for $\gamma = 0.08$ and $J = 0.13$ in the principal anomalous zone.