Ye olde inertia demonstration

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A simple quantitative analysis of the classic inertia ball demonstration explains why the lower string may break for ''jerks'' weaker than those that normally break the upper string, and why both strings may break—first the lower, then the upper.
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I. INTRODUCTION

A common introductory physics demonstration consists of a weight hung on a string with an identical string tied to the bottom of the weight (see Fig. 1).[1–4] A slow pull on the lower string breaks the upper string, but a fast pull breaks the lower string, leaving the upper intact. The qualitative explanation is that the inertia of the mass prevents the upper string from breaking when the lower is pulled quickly. To our knowledge, only a few papers have analyzed this demonstration quantitatively.[5–8] We discuss two surprising behaviors that we call *sequential breaking* and *anomalous breaking*.

In sequential breaking, both strings break—first the lower, then the upper. Sequential breaking occurs if, at the time that the lower string breaks, the mass has sufficient momentum to break the upper string. Sequential breaking was analyzed in Refs. 5 and 6. The former correctly analyzed sequential breaking for a special case. The latter generalized the analysis, but contains an error, which our analysis corrects.

In anomalous breaking, the lower string breaks for a pull slower than the pull required to break the upper string. Anomalous breaking, which was analyzed in Ref. 8, arises from the elastic behavior of the upper string, and it may be difficult to observe in practice. Our investigation offers an improved graphical method for understanding anomalous breaking.

II. THE MODEL

We choose a very simple model for our analysis.

(1) The force applied to the lower string starts at zero and increases linearly with time. This assumption means that the force applied to the lower string has the form

$$F_{\mathrm{appl}}(t)=\alpha t \quad (t>0). \tag{1}$$

If this force were applied to an isolated mass, the time rate of change of the acceleration would be proportional to the constant α. Thus, α directly measures the ''jerkiness'' of the applied force.[9]

(2) The strings are massless. This assumption means that F_{appl} can be considered to be applied directly to the mass m, and the spring constant of the lower string need not be considered.[10]

(3) The strings obey Hooke's law up to failure at tension T_0. Thus, the tension T_{up} in the upper string is

$$T_{\mathrm{up}}=k\Delta x \quad (T_{\mathrm{up}}<T_0), \tag{2}$$

where k and Δx are, respectively, the spring constant and the extension of the upper string.

Given this model, the equation of motion of the mass is the equation of motion for a mass m hung from a spring of spring constant k and driven by the force $F_{\mathrm{appl}}(t)$,

$$\ddot{x}+(k/m)x=(\alpha/m)t, \tag{3}$$

where x is the downward displacement from the equilibrium position of the mass hanging on the upper string,

$$x=\Delta x-mg/k. \tag{4}$$

For the standard initial conditions from rest at $t=0$, the solution is

$$x=(\alpha/k)[t-\sqrt{m/k}\sin(\sqrt{k/m}t)] \quad (t>0). \tag{5}$$

The tension T_{low} in the lower string is simply the applied force,

$$T_{\mathrm{low}}=F_{\mathrm{appl}}(t)=\alpha t. \tag{6}$$

The tension T_{up} in the upper string is

$$T_{\mathrm{up}}=mg+kx=mg+\alpha[t-\sqrt{m/k}\sin(\sqrt{k/m}t)]. \tag{7}$$

When $x=0$, the tension in the upper string is mg, the weight of the hanging mass. We call this tension the *static loading*.

III. ZONE BOUNDARIES

Given our model, we want to distinguish quantitatively between the upper string breaking first and the lower string breaking first. The boundaries between these two outcomes are *zone boundaries*. At the zone boundaries, the tensions in the two strings reach T_0 simultaneously. So, we set

$$T_{\mathrm{low}}=T_{\mathrm{up}}=T_0. \tag{8}$$

We eliminate the two string tensions and the time t from Eqs. (6) to (8) and obtain[11]

$$\frac{\sin(1/J)}{(1/J)}=\gamma, \tag{9}$$

where γ is the normalized static loading,

$$\gamma=mg/T_0, \tag{10}$$

and J is the normalized ''jerk,''

$$J=\alpha/(T_0\sqrt{k/m}). \tag{11}$$

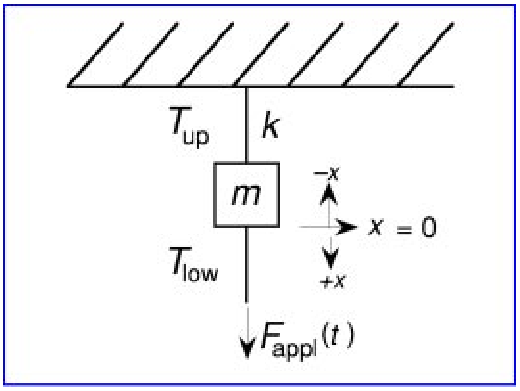

Fig. 1. Mass suspended on a string with a "jerky" force.

The resulting zone boundaries are shown as solid curves in Fig. 2, which plots γ against J.[12] For a particular value of γ, the value(s) of J satisfying Eq. (9) can be found by drawing a horizontal line on Fig. 2 at the appropriate height. For sufficiently small values of γ, Eq. (9) has multiple solutions that give rise to *anomalous zones*. For values of γ and J within these zones, anomalous breaking will occur. There is one such zone (the principal anomalous zone) for $0.071<\gamma<0.128$, two for $0.049<\gamma<0.071$, etc. These values of γ appear in Ref. 8, but Fig. 2 and Eq. (9) make the calculation and interpretation of these values much clearer. (Figure 2 is similar to Fig. 1 in Ref. 6, but the anomalous zones were overlooked there.)

As an example of anomalous breaking, consider $\gamma=0.08$. Because $J=0.34$ satisfies Eq. (9) when $\gamma=0.08$, the lower string will break for $J>0.34$. This result is expected for a fast pull on the lower string. The conventional qualitative analysis suggests that the upper string will break for all $J<0.34$, but this result is not exactly what our model predicts. For $\gamma=0.08$, $J=0.15$ and $J=0.12$ also satisfy Eq. (9), that is, the lower string breaks for $0.12<J<0.15$ as well as for $J>0.34$. Figure 6 shows a plot of the tensions for $\gamma=0.08$ and $J=0.13$; in Fig. 2 this point is labeled "Fig. 6."

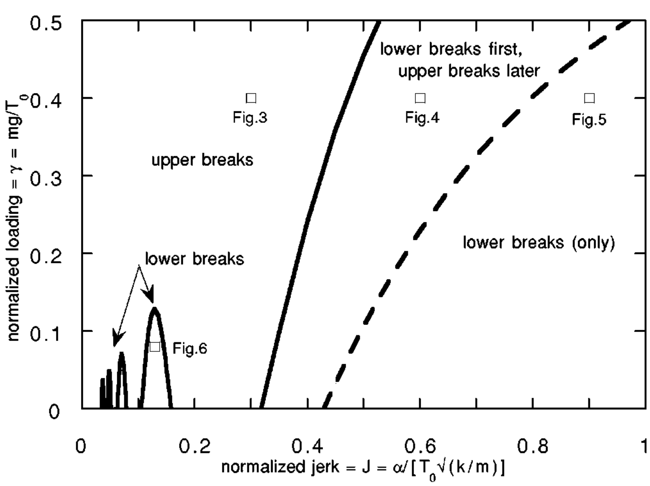

Fig. 2. Plot of the normalized static loading against the normalized jerk parameter showing the zones in which the upper or lower string breaks first. The dashed curve is the boundary for sequential breaking. The square symbols locate the parameter values used for the time plots in Figs. 3–6.

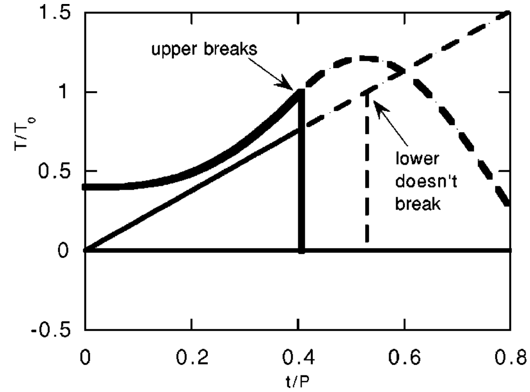

Fig. 3. The tensions in lower and upper strings for $\gamma=0.4$ and $J=0.3$, in the normal low-jerk zone. The respective tensions are normalized to the breaking strength T_0. The time axis is normalized to the free period of the mass on the upper string, $P=2\pi\sqrt{m/k}$. The heavier curve is the tension in the upper string.

IV. SEQUENTIAL BREAKING

When the lower string breaks, the mass may have sufficient momentum to cause the upper string to break later. From Eq. (6), the lower string breaks at the time $t_1=T_0/\alpha$. If we use Eq. (5) and its derivative, we can evaluate the mass's displacement and velocity at t_1 and then calculate the total energy E. The maximum subsequent displacement, $x_{\max}$, of the mass can then be found from

$$E=\tfrac{1}{2}kx_{\max}^2, \tag{12}$$

and it can be compared with the displacement $x=(T_0-mg)/k$ at which the upper string breaks [from Eq. (7) with $T_{up}=T_0$]. In terms of the normalized variables of Eqs. (10) and (11), the boundary for subsequent breaking of the upper string is given by

$$[1-(\sin(1/J)/(1/J))]^2+[(1-\cos(1/J))/(1/J)]^2 = (1-\gamma)^2. \tag{13}$$

This boundary for sequential breaking is shown as the dashed curve in Fig. 2.[13] An equation similar to Eq. (13) appears in Ref. 6, but contains an error;[14] consequently, the boundary for sequential breaking is shown incorrectly in Fig. 1 of Ref. 6.

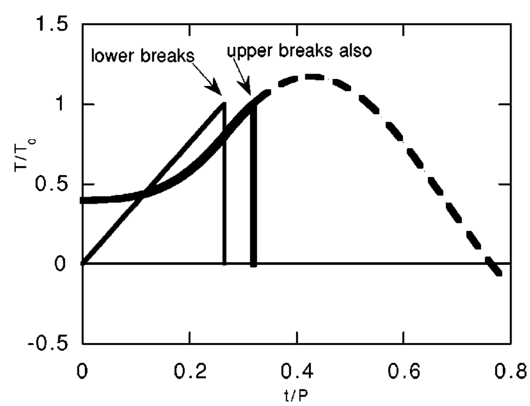

Fig. 4. String tensions for $\gamma=0.4$ and $J=0.6$ in the sequential breaking zone.

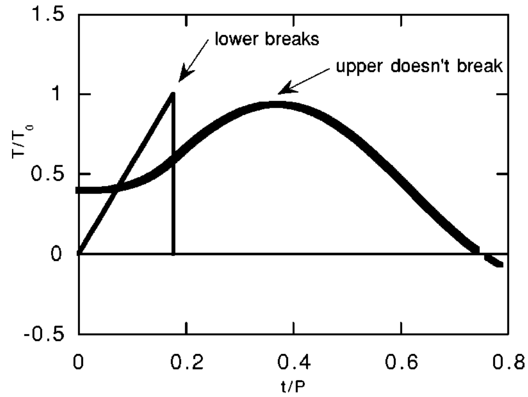

Fig. 5. String tensions for $\gamma = 0.4$ and $J = 0.9$ in the normal high-jerk zone.

As an example of sequential breaking, consider $\gamma = 0.4$. Then $J = 0.47$ satisfies Eq. (9), and $J = 0.80$ satisfies Eq. (13). This solution means that the lower string will break for all $J > 0.47$; and for $0.47 < J < 0.80$, the upper string will break after the lower string breaks. Figure 4 shows a plot of the tensions for $\gamma = 0.4$ and $J = 0.6$; in Fig. 2 this point is labeled "Fig. 4."

V. PLOTS OF THE TENSION

Figures 3–6 are time plots of the tensions in the two strings. The parameter values were chosen to represent the various zones in Fig. 2. The heavier curve is the upper string's tension—before the lower string breaks, this tension is T_{up} from Eq. (7); after the lower string breaks, it is the tension for the free oscillation of the mass starting with the position and velocity at the time of the break. The lighter curve is the lower string's tension from Eq. (6). The dashed curves are extrapolations showing what would have happened if the first string had not broken. (Negative tensions are, of course, not possible in real strings.)

With the exception of Fig. 6, these figures show breaking before the upper string has completed a cycle of its oscillation. Figure 6 shows that, for anomalous breaking, the lower string breaks after the upper string has completed more than a cycle, and the upper string never breaks.

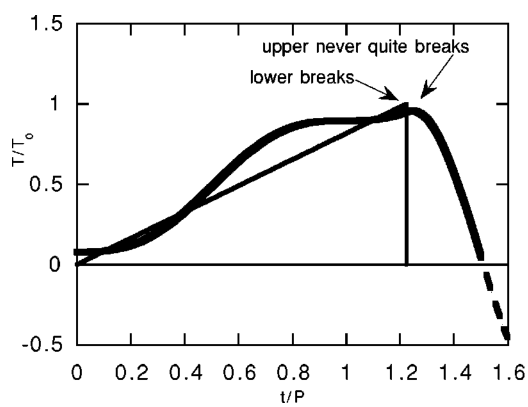

Fig. 6. String tensions for $\gamma = 0.08$ and $J = 0.13$ in the principal anomalous zone.

VI. CONCLUSIONS

Our simple model predicts anomalous breaking (shown in Fig. 6) in which the lower string breaks "on the bounce" for some combinations of light static loading ($\gamma < 0.128$) and jerks weaker than those that normally break the upper string. As discussed in Ref. 8, this behavior may be difficult to observe in the laboratory—even if the "jerky" force is better controlled than in the typical lecture demonstration.

For intermediate values of jerk, the model predicts sequential breaking (shown in Fig. 4) in which the upper string breaks after the lower string has broken. Sequential breaking may be more easily observed in the laboratory. Indeed, if an instructor is not forewarned, the double breaking might be an unwanted surprise in a lecture demonstration!

One can imagine building some sort of mechanical device to provide well-controlled, reproducible "jerky" forces (or displacements[10]) to demonstrate these two surprising and subtle behaviors.

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[3]*The Video Encyclopedia of Physics Demonstrations* (The Education Group & Associates, Los Angeles, CA, 1992), Disc 2, Chap. 5, Demo 02-13, "Inertia Ball."
[4]This demonstration is described at ⟨faraday.physics.uiowa.edu/mech/1F20.10.htm⟩ and ⟨www.physics.brown.edu/Studies/Demo/solids/demos/1f2010.html⟩.
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[9]T. R. Sandin, "The jerk," Phys. Teach. **28** (1), 36–40 (1990).
[10]The agent supplying F_{appl} is assumed to move in such a way as to produce the linear-ramp tension in the lower string. In Ref. 8 we considered an alternative model in which the lower end of the lower string is displaced linearly with time, producing a more complicated $T(t)$ in that string, and adding a dependence on the elastic constant of the lower string. Note also that the two strings or wires of the same material will have different spring constants k unless they are of the same length.
[11]A function of the form $\sin(u)/u$, familiar from the theory of single-slit diffraction, is often called the sinc function and is the spherical Bessel function of order zero.
[12]We thank John S. Wallingford (private communication) for suggesting this form of display. See http://www.smnet.net/jwally/.
[13]Like Eq. (9), Eq. (13) also has multiple solutions for small γ. There are narrow strips, just within the anomalous zone boundaries in Fig. 2, within which the sequential breaking occurs. These strips are too narrow to show in the figure and are unlikely to be observable experimentally.
[14]The proper equation is given in Ref. 5 for the special case of $\gamma = 0.5$. Reference 6 attempts to generalize to arbitrary γ and gives an equation equivalent to our Eq. (13), but with γ^2 rather than $(1-\gamma)^2$ on the right-hand side.