Which String Breaks?

By Mark A. Heald and George M. Caplan

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In *The Physics Teacher* for October 1995 (p. 478) Martin Gardner called attention to the well-known demonstration in which a large mass is suspended by a light string, with a similar string hanging from the mass (Fig. 1). If you pull slowly on the lower string, the upper string breaks. But if you pull rapidly, the lower string breaks. Most introductory textbooks give some version of this problem in qualitative form,[1,2] but we are not aware of a quantitative treatment that shows what parameters determine which string breaks.

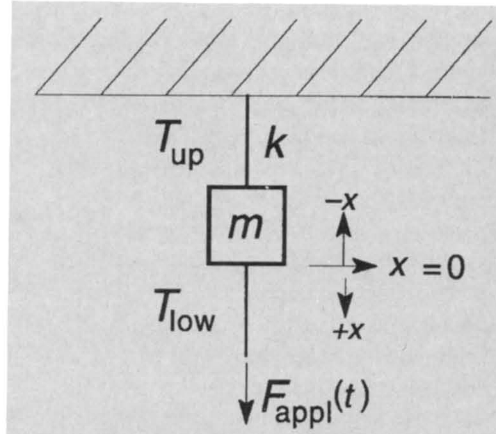


Fig. 1. Mass suspended on a string, with "jerky" force applied to lower string.

To quantify the problem, we must introduce two models: (I) the time-dependence of the applied force, and (II) the elastic properties of the strings. We will choose the simplest possible model in each case, recognizing that each model represents only a first approximation to the real world. The models are:

> *I. The force applied to the lower string will be assumed to increase linearly with time,*

$$F_{appl}(t) = \alpha t \quad \text{for } t > 0 \tag{1}$$

If such a force were applied to an isolated mass, there would be a constant time-rate-of-change of acceleration. This third derivative of displacement is often called the "jerk"[3] and is consistent with the use of the word in everyday language. So we can say that the constant coefficient α measures the magnitude of the jerk.

> *II. Each string will be assumed to be massless and to obey Hooke's law up to failure at the tension $T = T_0$ (Fig. 2). That is, the relation between the tension and the extension Δx of the string is given by*

$$T = k\,\Delta x \quad \text{for } T < T_0 \tag{2}$$

Because the lower string is massless, the tension is the same all along it and is equal to the applied force. That is, F_{appl} is effectively applied directly to the mass m. The resulting stretching of the lower string does not enter the analysis directly.

So long as neither string has broken, the equation of motion of the mass is

$$ma = F_{appl}(t) - kx \tag{3}$$

where x is the downward displacement of the mass from its equilibrium position as it hangs on the upper string. Substituting $F_{appl} = \alpha t$ and $a = d^2x/dt^2$, and rearranging, we obtain the differential equation of motion as

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{\alpha}{m}t \tag{4}$$

This is the well-known differential equation for driven simple harmonic motion. The solution consists of the sum of a complementary function [which is the solution to $d^2x/dt^2 + (k/m)x = 0$], plus a particular integral determined by the driving term. Thus the solution to Eq. (4) is of the form

$$x = A \sin(\sqrt{k/m}\, t + \phi) + (\alpha/k)\, t \tag{5}$$

where A and ϕ are constants of integration, to be determined by the initial conditions of the system. Since we assume that $x = 0$ at $t = 0$, the phase is $\phi = 0$. The value of the amplitude A is determined by the second initial condition, $dx/dt = 0$ at $t = 0$, from which

$$A = -\frac{\alpha}{k}\sqrt{\frac{m}{k}} \tag{6}$$

The complete solution of Eq. (4) satisfying our initial conditions is thus

$$x = \frac{\alpha}{k}\left[t - \sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)\right] \tag{7}$$

The tension T_{low} in the lower string is simply the applied force,

$$T_{low} = \alpha t \tag{8}$$

The tension T_{up} in the upper string is

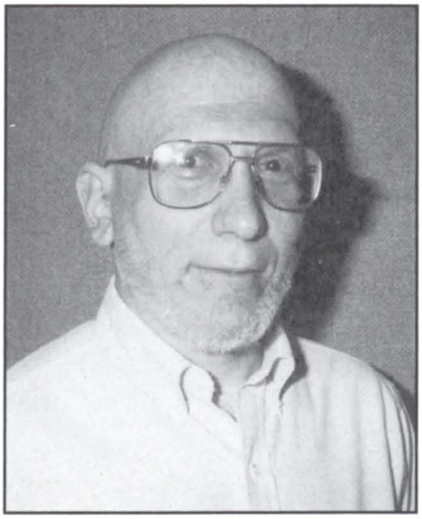
$$T_{up} = mg + kx = mg + \alpha\left[t - \sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)\right] \tag{9}$$

Figure 3 plots the tensions in the two strings as a function of time. In this figure we have assumed that the "dead weight" term mg in Eq. (9) is negligible compared to the breaking tension T_0 — that is, that the upper string was not stretched anywhere near its breaking strength initially. (Note that the extension x is measured from the *stretched* position of the mass hanging on the upper string.)

The two curves intersect at the tensions labeled T_1, T_2, T_3, etc., which occur at the times such that

$$\sqrt{\frac{k}{m}}\,t = \pi, 2\pi, 3\pi, \text{etc.} \quad \text{(for negligible } mg) \tag{10}$$

Meanwhile, the breaking tension T_0 corresponds to some value on the vertical axis of Fig. 3. If T_0 is less than T_1, failure occurs in the lower string first. If T_0 lies



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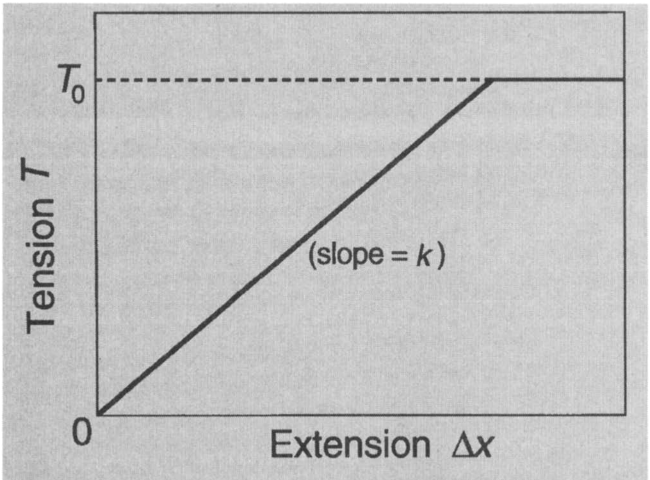

Fig. 2. Simplistic model of string elasticity: the string breaks when the tension reaches T_0.

between T_1 and T_2, failure occurs in the upper string first. Postponing for the moment the case where T_0 lies above T_2, we look at the condition where $T_0 = T_1$, at the boundary between the two lower zones:

$$T_0 = \alpha_1 \sqrt{\frac{m}{k}}\,\pi \quad \text{or} \quad \alpha_1 = \frac{T_0}{\pi}\sqrt{\frac{k}{m}} \tag{11}$$

If the jerk-parameter α is greater than this value α_1, then T_1 is greater than T_0 and the lower string breaks. If it is less than this value (but not too much less?), the upper string breaks.

This boundary condition can be put in more physical terms by expressing the oscillation frequency of the mass on the (upper) string in terms of its period, $P = 2\pi\sqrt{m/k}$, and the jerk value in terms of the "time to failure," $\tau = T_0/\alpha$. Then Eq. (11) becomes

$$\frac{T_0}{\alpha} = \pi\sqrt{\frac{m}{k}} \quad \text{or} \quad \tau = \frac{P}{2} \tag{12}$$

That is, the applied force must reach the failure value in less than one half-period of the oscillation for the lower string to break.

In Fig. 3, the zone for $0 < T_0 < T_1$ corresponds to large α (i.e., for a strong jerk, the lower string breaks), while the zone for $T_1 < T_0 < T_2$ corresponds to smaller α (for a weak jerk, the upper string breaks). This relationship is exactly what one expects physically, arising from the inertia of the mass. However, if T_0 is greater than T_2 (corresponding to an even *weaker* jerk), Fig. 3 suggests that there can be alternating *anomalous* zones where the *lower* string breaks first—quite contrary to one's physical understanding of the situation. There are several reasons why our simplistic analysis fails when T_0 is relatively large on the scale of the figure:

1. Most importantly, under the usual conditions of the demonstration, the static-weight term mg in Eq. (9) is *not* negligible compared to T_0. A significant mg simply translates upward the wiggly curve in Fig. 3. This shift moves the intersection points (T_1 down, T_2 up, etc.), thus reducing the width of the zones where the lower string breaks, and increasing the width of zones where the upper string breaks. (Indeed, when mg is greater than $\alpha\sqrt{m/k}$, the wiggly curve is shifted so far that the upper string always breaks first.) We discuss this effect further below.

2. We have neglected the damping of the oscillation that arises when the string is imperfectly elastic (physically the oscillation is excited when the jerk begins at $t = 0$). For most real-world strings, the oscillation of T_{up} shown in Fig. 3 will damp out quickly with time. Putting this effect together with the mg shift can allow the T_{up} curve to lie above the T_{low} line everywhere above the (new, shifted) T_1 intersection, and the upper string breaks for all T_0 values greater than T_1.

3. The sudden transition from Hooke's law to failure, assumed in Fig. 2, is unrealistic. In most cases this sharp "corner" will be rounded. The rounding would correspond to a lower k value, and hence a longer period of oscillation of T_{up} as the tensions approach T_0; failure of the upper string is likely to occur well before the "T_2" intersection is reached.

In the standard demonstration, it is the jerk value α that is the *variable* quantity, with mg, T_0, and k held *fixed*. But in Fig. 3 the effect of varying α (which is hidden in the slope of the curves), while holding the other parameters constant, is not easy to see. In particular, it may be quite reasonable to neglect mg when the failure value T_0 occurs below the T_1 intersection, but *not* reasonable when the failure is in the vicinity of T_2 or above (in which case the T_{up} curve would need to be shifted). Also note that our primary interest is in the intersection points in Fig. 3, rather than a full time-dependent description. That is, we are concerned with the *boundaries* between one string or the other breaking, which occur at conditions such that

$$T_0 = T_{up} = T_{low} \tag{13}$$

Suppose that mg is a fixed fraction of the failure tension T_0, and let us call this constant ratio gamma,

$$\gamma = \frac{mg}{T_0} \tag{14}$$

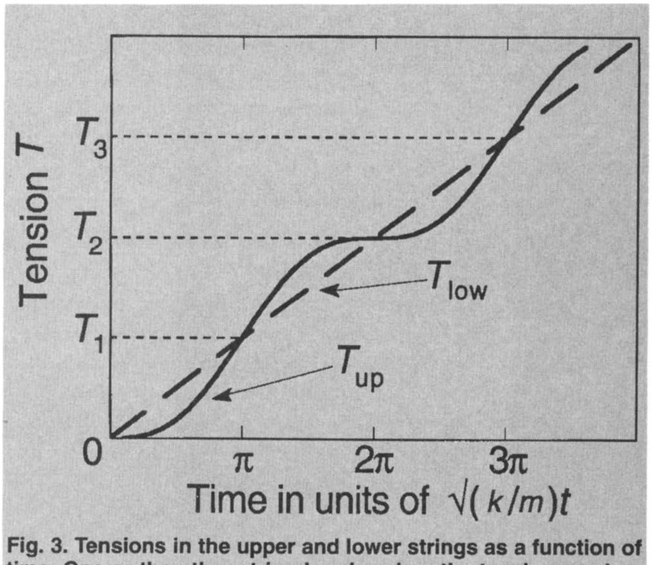

Fig. 3. Tensions in the upper and lower strings as a function of time. One or the other string breaks when the tension reaches the vertical coordinate equal to the failure value T_0.

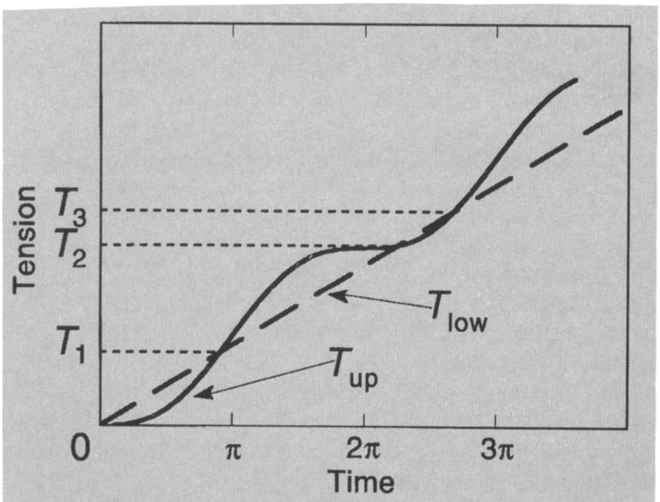

Fig. 4. Schematic distortion of Fig. 3, assuming the fixed ratio $\gamma = mg/T_0 = 0.1$. See text.

For the special circumstance that $T_0 = T_{up}$, we can now rewrite Eq. (9) as

$$T_{up} = \gamma T_{up} + \alpha \left[t - \sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)\right]$$

$$T_{up} = \left(\frac{1}{1-\gamma}\right) \alpha \left[t - \sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)\right] \qquad (15)$$

Figure 4 is a replot of this special T_{up} (for $\gamma = 0.1$), together with the T_{low} of Eq. (8). In this modified figure, only the *intersection points* [where Eq. (13) applies] are quantitatively meaningful. (In between intersections, the curves are meaningful only to the extent of indicating which string breaks in that zone.) We have a discrete set of intersection points (boundaries), depending upon whether the actual T_0 coincides with T_1, or T_2, or possibly T_3, etc. [The values of T_1, T_2, etc., are now modified from those of Eq. (10) by the nonzero value of mg.]

This modified plot in Fig. 4 shows that, for the common case with fixed ratio $\gamma = mg/T_0$, the anomalous zone between T_2 and T_3 is substantially reduced in size (and, all the more so, the higher-order anomalous zones between T_4 and T_5, etc.). In particular, when the gamma ratio is 0.128, the T_{up} curve just touches the T_{low} line, and the intersections T_2 and T_3 coalesce to a single point. That is, for $\gamma > 0.128$, there are no anomalous zones, and the upper string breaks for all values of jerk α less than the value corresponding to T_1 [Eq. (11) modified for nonzero mg]. The only possibility of seeing failure of the *lower* string "on the bounce" would be for very lightly loaded strings, $mg < 0.128T_0$. And this value would be still lower yet in the likely presence of damping, and of reduction of the effective spring constant k close to failure.[4]

Our "simplest model" of the applied force, Eq. (1), has the advantage that the elastic property (k value) of the lower string does not enter the analysis, but the practical defect that the bottom of the lower string (i.e., the point of application of the force) has an oscillatory component of motion. Arguably, a more realistic alternative is to assume that the *position* of the bottom of the lower string changes linearly with time. For this model, because of the variable extension of the lower string, the *applied force* now has an oscillatory component superposed on the linear ramp. The elastic properties of both strings are now required; if the strings are of similar material, the value of their Young's modulus will be the same, but their "spring-constant" values, k_{up} and k_{low}, will now be in proportion to their respective lengths. Thus the analysis is encumbered by the second k value (or a constraint on the ratio of lengths). When the analysis of this alternative model is carried through, it turns out that the straight line for T_{low} in Fig. 3 becomes oscillatory, but in the opposite sense to the wiggles of T_{up}.[5] Nevertheless, although the analysis becomes somewhat more complicated, the qualitative behavior remains essentially the same. We note that this choice—of whether the applied force is linear in time (and its point of application wiggles), or the point of application moves linearly in time (and the magnitude of force wiggles)—depends physically upon the mass (and perhaps other properties) of the forcing agent. As noted at the beginning, it is hard to construct a quantitative model that represents realistically all the details of a typical experimental situation.

References

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2. G.D. Freier and F.J. Anderson, *A Demonstration Handbook for Physics*, 2nd ed. (AAPT, College Park, MD, 1981), p. M-16, Demonstration Mc-2.
3. T.R. Sandin, "The jerk," *Phys. Teach.* **28** (1), 36–40 (1990).
4. The anomalous zone between T_2 and T_3 remains present, but the zone between T_4 and T_5 disappears, for $\gamma > 0.071$. Experimenters might try to observe this "anomalous" breaking of the lower string for weak jerk by working with $0.071 < \gamma < 0.128$. Reproducibility and low damping suggest using fine, brittle wires, rather than fibrous string or thread.
5. The periodicity of oscillation also changes because the two strings now work in "push-pull." The effective spring constant k in the argument of the sine function in Eqs. (7) and (9) becomes $k_{up} + k_{low} \approx 2k_{up}$.